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Business Cycles in the Phillips Machine

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Abstract

Over the summer of 2003, the author undertook the refurbishment of the Cambridge Phillips Machine with help from technicians in the Cambridge University Engineering Dept and with advice from economists. The Machine now works and - moreover - is safe to work with. The Machine has since been used to give numerous working demonstrations to a wide variety of audiences from schoolchildren to distinguished economists.

This paper describes some of the standard experiments that can be conducted on the Machine. Also described are more recent simulations which attempt to demonstrate the possibility of generating business cycles - of both linear and nonlinear Hicksian types - from the basic accelerator-multiplier system.

1 Introduction

The Faculty of Economics at Cambridge University possesses a Mark II Phillips Machine. It is kept in a specially-built, locked, glass-fronted cabinet in the Meade Room of the Austin Robinson Building on the Sidgwick site in Cambridge to protect it from accidental damage and souvenir pilfering. It had received some restorative work in the early 1990's but was inoperable, largely for safety reasons associated with large voltages applied directly into the water.

With objects of historical significance there is always a dynamic tension between conservation and restoration. Given the existence of the beautifully-conserved Machine in the Science Museum in London, the Cambridge Faculty decided to allow their Machine to be restored to full working order, provided there was minimal intrusion on the original workings. Professor William Brown was central to all such discussions, his interest fostered in part by his father's role in funding the construction of Phillips and Newlyn's original Mark I prototype at the LSE (and which is now carefully preserved - but inoperable - at the Business School of Leeds University).

The restoration took place over the summer of 2003, which coincided happily with the centenary celebrations of the Economics Tripos at Cambridge. The project was coordinated by myself, with much of the work being done by the technical staff at Cambridge University Engineering Department.

There were numerous obstacles to be overcome - financial, transportational, and even just understanding how everything was supposed to be connected - but the key element was electrical: three water level sensors had 240V electrodes dipping directly into the water in obvious contravention of modern safety regulations. In the only intrusion on the original workings, these were stepped down to 20V by Mr Terry Hoppitt, and the additional electrical circuitry was housed neatly in a discrete box at the rear of the Machine.

Once operable, the author reconnected all the various moving parts, and Prof Brian Henry then kindly demonstrated the three basic simulations that will be described below. Its inaugural demonstration was at the Cambridge Centenary celebrations - an event that celebrates not merely the (albeit remarkable) local achievements - but the occasion on which Economics first emerged from under the wing of Moral Philosophy as a subject worthy of study in its own right. Attendees including a distinguished audience of alumni, including Sir Edward George. The Machine worked perfectly, demonstrating a balanced budget multiplier (for the two cases of fixed interest rate and fixed quantity of money) and a simple use of the accelerator.

2 Machine overview

The mechanical and hydraulic workings of the Machine are described in Phillips' original 1950 *Economica* paper. Here we describe a few features necessary to understand the later experiments.

The first feature to note is the outlet slot on the side of the main tank at the base of the Machine, out of which flows National Income Y. Phillips seeks to emulate the linear relationship between the quantity of money M_1 and Income Y that is basic to the quantity theory of money. He achieves this via a feature known as a Sutro weir. Weirs typically have a nonlinear relation between head and discharge, but a Sutro weir is a specially-shaped slot that tapers with height in such a way that the outflow is **linearly proportional** to the head behind the slot. This tapering, to first approximation, goes as the reciprocal of the square root of the height. Via this device, Phillips replicates the equation $M_1 = PY$, where the proportionality constant P can be identified as the **circulation period**. (Note: P is not Prices.)

The Income flow Y leaving the Sutro weir is then pumped to the top of the Machine where, inevitably, it is immediately taxed. The Income Y enters a small enclosure which maintains a constant head over the Taxation T waterfall to the upper left of the Machine. Like many of the flows on the Machine, the Taxation flow is regulated via a graph-driven slider which partially blocks a horizontal slot through which the water falls. The graph is a thin piece of plastic containing a sloping (and possibly nonlinear) slotted hole. The graph can move vertically, powered - in this case - via a set of pulleys which connect it to a large float on the main M_1 tank at the base of the Machine. Since, via the quantity theory of money (and the Sutro weir), the main tank water level is a proportional measure of Income Y, the Taxation graph rises and falls in proportion to Y. Phillips has arranged the constant head above every such waterfall so that the linear scale factor on the horizontal slot is identical to the vertical scale implicit in the Sutro weir. Brilliantly then, the axes on all the plastic graphs have equal scales. If the plastic

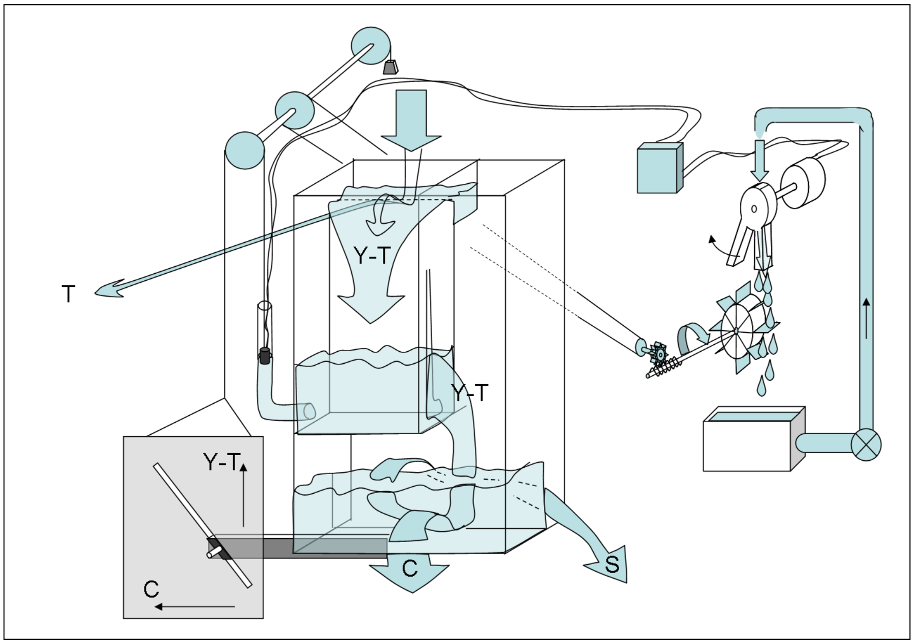


Figure 1: The hydromechanical representation of the Multiplier: After Tax is removed, the net income flow $Y - T$ is first measured with a Sutro weir and manometer and a remarkable contrivance at the rear of the Machine drives the vertical motions of the Propensity to Consume graph in direct proportion to $Y - T$.

graph is linear with a slope of 3:1 then the (marginal) taxation rate is 33 percent. (In all experiments described here, however, the slotted taxation graph is omitted such that the waterfall length - and hence the total annual taxation revenue T - remains constant throughout).

The Government having taken its tax, the residual income $Y-T$ then spills over and falls down the centre of the Machine. A key component of the Machine is the graph of Propensity to Consume. Again, it consists of a plastic sheet with a slotted hole in which rides a pin connected to a slider which controls the exposed length of a waterfall - in this case the Consumption waterfall. The fundamental difference between this graph and most others on the Machine is that the vertical motions of the graph cannot be powered by a large float on a stock (such as M_1 at the base of the Machine). The vertical axis on the Propensity to Consume graph is Net Income ($Y - T$) and this is a flow, not a stock. In order to drive the graph up and down based on the **flow** $Y - T$ Phillips has had to resort to an ingenious but remarkably circuitous contrivance illustrated in Fig. 1. The details of this are of no interest to economists but will amaze any readers who are engineers.

First, Phillips measures the flow $Y-T$ by feeding it into a small chamber (of essentially negligible volume) whose only function is to measure the flow rate. The outflow from this chamber is via a Sutro weir identical to that at the base of the Machine. The height of water in the chamber is - due to the Sutro linearity - proportional to the outflow $Y - T$, and is measured by a small manometer tube at the side of the chamber, into which a pair of electrical contacts are suspended via a pulley above. This pulley also raises and lowers the Propensity to Consume graph, thus if the electrical contacts can track the water surface in the manometer, then the graph will rise and fall in direct proportion to the flow $Y - T$. This heavy graph is first counter-weighted, and its motions are controlled by a remarkable system located at the back of the Machine. Here, a secondary pump feeds water into the top of a toggle-waterwheel system. The flow in this hidden part of the Machine has no economic meaning - it is there merely to power the lifting of the Propensity to Consume graph. If the electrical contacts are submerged in the manometer, a current flows which is fed to a small motor which drives a hollow two-legged "toggle" either to the left or the right. The falling secondary water thus is directed down one leg of the toggle to turn a waterwheel. This in turn drives - via a worm-gear, cog, pulley and a long "drive belt" - the upper pulley supporting the graph. If the electrical contacts are in the manometer water the system raises them (and thus lowers the graph). Once out of the water, the mechanism reverses - the toggle is pushed the other way such that the water flows down its other leg to drive the waterwheel in the opposite direction, and - via the worm gear/cog/pulley/belt/pulley - thereby lowers the electrical contacts back into the manometer water. By this extraordinary method, the electrical contacts are kept at the manometer water surface, and the Propensity to Consume graph is thereby moved vertically in direct proportion to the flow $Y - T$.

In the Machine as found, the electrical contacts in the manometer tube carried 240 Volts directly into the water. There were two other places on the Machine (one for the main pump driving Income Y and one regulating foreign sector transactions [1]) which used such a 240V-into-water arrangement. The only physical intrusion on the original workings of the Machine was the stepping down of these voltages to Health and Safety compliant levels of 20 Volts. The through-the-water current of the electrical manometer sensor was then insufficient to reliably drive the back-of-Machine motors, and so a relay was added to boost the low current signals from the sensor.

The Consumption waterfall again has a small constant-head enclosure above it with an overspill weir such that the residual flows $(Y - T) - C = S$, Savings, spill over and are directed out to the banking sector to the right-hand side of the Machine.

This remarkable hydro-mechanical arrangement forms the core of Phillips' representation of the Keynesian multiplier.

The representation of the banking sector as the supply-and-demand of a stock of loanable funds is described in Phillips' original paper. A float on the bank water level drives a slotted graph controlling Investment outflows from the bank. Investment behaviour is thereby related to the quantity of money held in the bank and thence - via a curved end-wall in the banking tank representing the liquidity preference function - to interest rates. A higher graph - powered by the same float on the bank water level - can adjust the left-hand side of the Consumption waterfall such that Saving behaviour can also be related to interest rates. With one minor exception, these graphs were not used in the experiments that follow.

[1] Regulation of the foreign sector flows required not only a high voltage water sensor, but an additional back-of-Machine system of motor/toggle/waterwheel/wormgear/cog/pulley arrangement, powered by water falling from the first. Figure 1 thus shows only half of the arrangement behind the Machine.

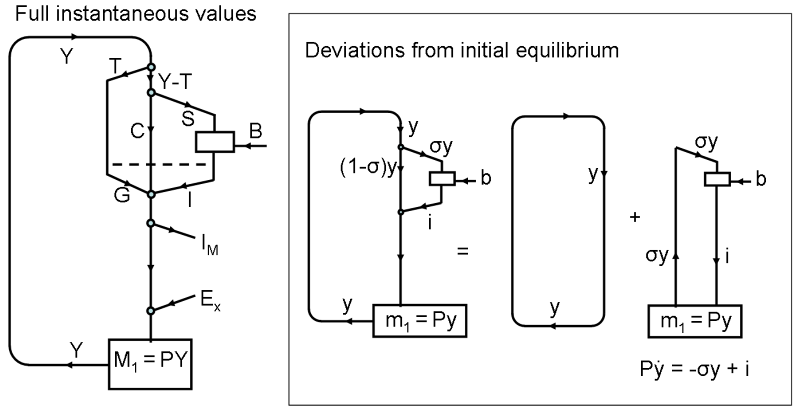


Figure 2: Derivation of the basic multiplier equation. At the left, the full instantaneous flow diagram is shown. To the right, the deviations of the flows from initial equilibrium are shown (when the government and foreign sector flows are kept constant). These marginals are decomposed into the sum of two flows. Applying the basic conservation law "Rate of change of storage = inflow - outflow" to the lower tank in the final figure leads immediately to the multiplier equation $P\dot{Y} = -\sigma Y + I$

The experiments that follow involve the "Multiplier-Accelerator" model. The hydromechanics of the multiplier has now been explained and that of the accelerator will be described shortly.

3 The Multiplier-Accelerator Model

The multiplier-accelerator model involves the interaction of Consumption C, Savings S, Investment I and Income Y. It can can best be explored by shutting down most other features of the Machine: the foreign sector flows can be turned off and the government sector can be set to a balanced budget with constant taxation and expenditure.

When interest rates are fixed there is only one control graph of interest - the Propensity to Consume - whose slope dictates what fraction of marginal Income after Taxation $(Y - T)$ is consumed, the residual going as Savings $S = \sigma(Y - T)$ to the bank.

The multiplier equation

$$P\dot{Y} = -\sigma Y + I \tag{1}$$

is derived from the Machine flows in Fig. 2. Here, P is the circulation time and σ is the marginal propensity to save.

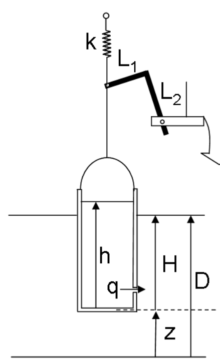
In its Machine realisation, the accelerator is a small bucket supported by a spring and floating in the main lower tank. A rise in the main tank water level will cause the bucket to rise and its support wire, via the bell-crank above it, will further open the investment valve. The accelerator is thus a model of expectations: as investors observe the economy to be rising their expectations of good times ahead will - in this model - cause them to increase their levels of investment.

The bucket has an adjustable hole such that the bucket does not perfectly track the main tank water level, it having a tendency to revert slowly to the original equilibrium position. This feature introduces a lag between the increase in main water level (a measure of income growth $\dot{Y}$) and the subsequent induced investments I. The resulting equation for the accelerator is derived in Box 1. Expressed in economic rather than hydraulic units, this is

$$\gamma\dot{I} = \beta\dot{Y} - (I - \Delta I) \tag{2}$$

where all variables are expressed as marginals relative to the initial equilibrium, and ΔI is the investment shock applied at the start of the run (since the accelerator is connected **after** the shock is applied).

Box 1: Accelerator mechanics



Vertical equilibrium requires:

$$\gamma_w A(h - H) = kx_1 \tag{3}$$

where γ_w is the unit weight of water, A is the plan area of the bucket, k is the spring stiffness and x_1 is the downward extension of the spring from its equilibrium position when $h = H$ and $z = z_0$.

Orifice flow out of the hole is governed by a square-root power law with respect to the head difference $h - H$. If this is linearised over the range of interest, it may be approximated as $q = c(h - H)$. Conservation requires $q = -A\dot{h}$. These combine to give $(k/\gamma_w A)\dot{x}_1 + \dot{H} = -x(c/A)(k/\gamma_w A)x_1$. The bell-crank gives $x_1 = (L_1/L_2)x_2$ and $x_2 = -K(I - \Delta I)$ where K is a factor converting economic units to mechanical units (such that $D = KY$, for example). Also $H = D - z$ thus $\dot{H} = \dot{D} - \dot{z}$ and $\dot{z} = -\dot{x}_1 = -(L_1/L_2)\dot{x}_2$. Substitution leads to the final accelerator equation

$$\gamma\dot{I} = \beta\dot{Y} - (I - \Delta I) \tag{4}$$

where $\gamma = ((1 + \phi)/\phi)(A/c), \beta = (L_2/L_1)(A/c)$ and $\phi = k/\gamma_w A$.

4 Standard Simulations

The three standard simulations currently performed in public demonstrations are those described in Phillips' 1950 *Economica* paper: the balanced-budget multiplier in the two cases of fixed interest rate and fixed quantity of money, followed by the fixed interest multiplier including the accelerator. Typical outputs are shown in Fig. 3. All simulations shut down the foreign sector and concentrate on the interaction between banking and the high street, with government taxation and expenditure balanced at a nominal figure of £1 billion per year each (Machine values).

An initial equilibrium is created with an income of £4B/yr split 1:2:1 between the government, consumption and savings/investment anabranches. The graph relating consumption to income after tax is set with a slope of around 2, (i.e. mpc = 0.5, mps = σ = 0.5). In the first demonstration, interest rates are pegged by keeping open the constant head device (the Royal Mint) at the side of the Machine. The equilibrium is disturbed by suddenly increasing investment from £1B/yr to £2B/yr. Standard multiplier theory states that, with interest rates fixed, the income Y should grow in a decreasing-exponential fashion, rising from the original £4B/yr to a final asymptote of £6B/yr. As can be seen from the Run 1 results of Fig. 3, the mechanical asymptote delivered by the Machine is approximately correct. Further detailed analysis (not presented here) also reveals that the time constant for the exponentially-decreasing growth is also captured well.

To save time, the second demonstration starts at the new £6B/yr equilibrium (with its 1:3:2 G:C:I split) and endeavours to return to the original 1:2:1 equilibrium by suddenly decreasing investment from £2B/yr back to its original £1B/yr. Before applying the shock, however, the constant head (Royal Mint) valve is closed, such that the Machine is now operating with a constant quantity of money. The investment outflow is also connected to its adjacent control graph which relates investment behaviour to interest rates. As the investment is suddenly cutback, the water level in the Bank begins to rise (because savings now exceed investment borrowing). This rise in the bank water level causes a decrease in interest rates (as described in Phillips' text). Via the control graph below the Bank powered by the float on the Bank water, this fall in interest rates begins to re-open further investment flows, thereby eroding into the initial shock decrease. Investment, rather than staying at the post-shock level of £1B/yr, now begins to rises back up to around £1.3 billion. Via this process of 'crowding', income Y does not fall back down to £4B/yr along an identical (but inverted) decaying exponential curve. Instead it falls by less, the multiplier being smaller. Although further longer-term adjustments to interest rates may occur, the initial effect showing how 'crowding' mitigates the full impact of the basic multiplier is clear [2].

The final standard demonstration involves reverting to the original 1:2:1 equilibrium and repeating the first multiplier simulation, but with the accelerator connected.

[2] The author has conjectured that a similar experiment conducted purely on the foreign sector further down the Machine can reveal how changes to exchange rates will erode into the positive benefits that would be expected to accrue from the sudden fall in UK imports that accompanied the start of North Sea oil production. Economists I have discussed this with have disagreed with me (and with each other) on this point

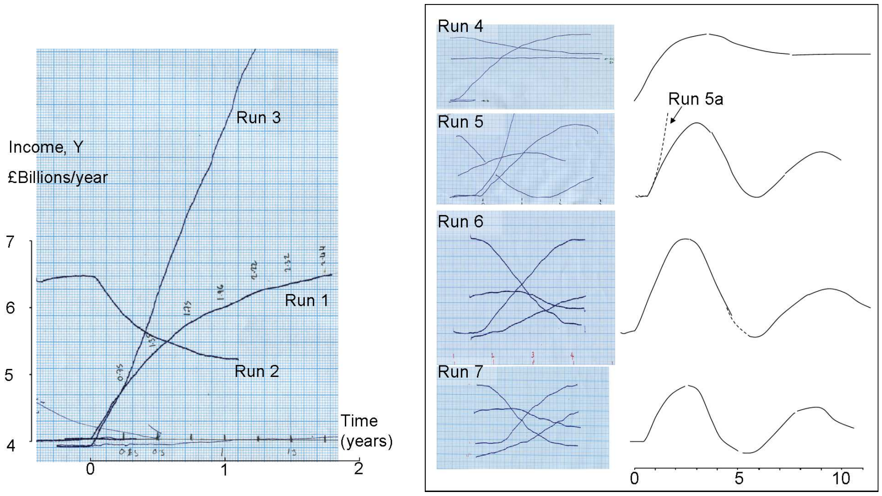


Figure 3: Left: A sample output graph from the pen plotter on the Machine, showing Income Y versus time, for the three standard demonstrations runs. Runs 1 and 2 show the multipliers at fixed interest rate and fixed quantity of money respectively. Run 3 is as Run 1, but with the accelerator attached to investment. Right: The raw Machine output for income Y for four linear responses to a small investment shock applied to the accelerator-multiplier system, together with representations having the time axis unwound (and compressed) to show the full response.

5 Cycles

The basic equations for the accelerator-multiplier model are developed in Fig. 2 and Box 1. The derivation differs only in style from that presented in Phillips' original paper. The resulting pair of coupled first-order ordinary differential equations that apply to all simulations in the remainder of the paper are:

$$P\dot{Y} = -\sigma Y + I \tag{5}$$
$$\gamma\dot{I} = \beta\dot{Y} - (I - \Delta I) \tag{6}$$

where flow deviations from initial equilibrium values are here (and henceforth) denoted by capital letters.

5.1 Damped and Undamped Linear Oscillations

Phillips' original 1950 *Economica* paper describes how accelerator-multiplier combinations can give rise to a damped linear oscillatory response, provided the parameters satisfy the inequality:

$$(\gamma\sigma + P - \beta)^2 < 4\gamma P\sigma \tag{7}$$

In a footnote, Phillips attributes such an oscillatory model to an early paper of Goodwin [1], and there were others - such as Hicks [3] - who had also explored such ideas. Further reference to the governing equations reveals that when the left-hand side of the parameter inequality (7) is zero, there is also the possibility of undamped linear oscillations. Given a small initial external shock, the income would then - in theory at least - oscillate forever, neither growing nor decaying.

The Phillips Machine can be set to undergo damped linear oscillations by adjusting the accelerator-multiplier used in the standard demonstration (Run 3, Fig. 3). The response to a small investment shock of four such set-ups is shown at the right of Fig. 3. The first simulation (Run 4) shows a quite heavily damped response and the lower three (Runs 5-7) show more lightly damped oscillatory responses. The second graph also shows a run (Run 5a) where the accelerator tuning is such that the response grows rapidly. This is similar to the ‘explosive’ response of Run 3. Mathematical simulation of such cases reveals that these are actually oscillatory divergent, rather than purely divergent. However, the first upswings are so large, rapidly reaching the limits of the Machine, such that the runs must be terminated before the subsequent vigorous downswings predicted by the theory can be observed.

Further fine-tuning of the machine parameters - particularly the accelerator orifice size and the bell-crank lever arm ratio - could no doubt be undertaken to home in further on the purely undamped case, where cycles neither grow nor decay. However, this was not pursued, largely by way of acknowledging Goodwin’s argument that if undamped oscillations require careful tuning of the parameters then they are not generic, and are unlikely to occur in reality. Instead, further attention was thus focussed on nonlinear cycles.

5.2 Nonlinear cycles

Hicks [3] and Goodwin [1] described how cycles in the accelerator-multiplier system might arise if an oscillatory divergent system were constrained to operate between upper and lower bounds. Goodwin drew the evocative analogy of an over-energetic pendulum bouncing forever back and forth between two walls. An upper bound on income was postulated to correspond to the condition of full employment. The lower bound was on investment, the argument being that gross investment could fall no further than zero, net investment then having a negative value corresponding to depreciation.

The plausibility for this ceiling/floor model over the previous linear cyclic models is that in order to obtain sustained oscillations, the parameters do not need to be finely-tuned to the condition of non-decaying, non-expanding oscillations. Instead, parameters need merely be such that the basic equilibrium system is unstable, a far more generic set of conditions.

Such a model can be simulated on the Phillips Machine. A smallish hole in the accelerator and a sizeable lever arm on the investment bell-crank can readily ensure the strongly divergent response (e.g. as per Run 3). A natural upper bound on income Y is provided by the Machine’s overflow weir on the main water tank, which overspills back into the holding tank at the rear of the Machine. (This is essentially nothing more than a safety mechanism to prevent the Machine overflowing and spilling water into the room, but it can also nicely serve as an upper bound). The Machine also has natural

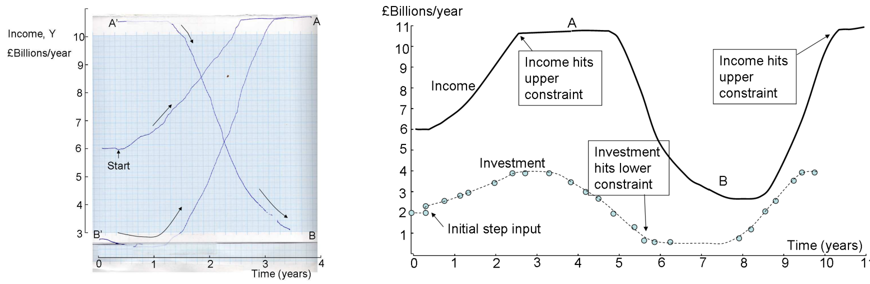


Figure 4: The Machine output for an occurrence of Hicks-Goodwin cycles. The raw data (left) is concatenated (right) to remove the time-axis rewinding. Also shown in the right-hand diagram are the corresponding (discrete) values of investment that were recorded.

lower bounds on investment occurring either when the gross investment waterfall is fully closed, or when the accelerator bucket grounds itself on the base of the main tank (whichever happens the sooner).

The very first attempt to explore such a model met with some success, adding credence to claims of genericity of the ceiling/floor model. The Machine output is shown in Fig. 4. After the initial small increment in investment, both income and investment rose rapidly, powered along by the combined action of the multiplier and accelerator. Income soon reached its maximum level, and the main tank began to overflow into the rear of the Machine. Over the course of a Machine-year on this ‘full employment’ plateau, investment fell rather slowly, reaching a point when both Y and I fell sharply, tumbling all the way down to the lower bound on investment. (The lower bound in this case was the accelerator bucket grounding at a gross investment level of around £0.6B/yr). Investment remained at this minimal level for around two years, whilst income fell ever less rapidly, with both eventually swinging upwards again and heading off rapidly to a second full-employment boom.

6 Accelerator Mechanics

Closer inspection of the accelerator mechanics reveals that the Machine undergoes a short transient and then converges immediately to a pure cyclic behaviour that it will undergo forever thereafter. Following the transient (consisting of two distinct phases) the subsequent cycle has four distinct phases.

During the initial transient boom, both Income and Investment rise until Income saturates its upper bound. It remains there whilst Investment falls on an exponentially-decaying trajectory until it reaches the criterion at which full Income can no longer be maintained, which the basic multiplier equation gives as $I = \sigma Y_{max}$.

Thereafter Income begins to fall rapidly and the Machine embarks on its pure cyclic behaviour. The “bust” phase eventually bottoms out when Investment reaches its lower bound. Income falls less rapidly and eventually turns around, both Income and Investment then embarking on a boom phase back up to the full-employment plateau. It will be shown that the conditions at the end of the boom plateaux are always identical, such that the system has arrived rather rapidly at its final pure cycle.

Between the upper and lower bounds, Income and Investment are governed by the coupled Phillips equations. The algebra simplifies if the marginals Y and I are defined relative to the final post-shock equilibrium $Y_{\mathrm{eq,fin}} = Y_{\mathrm{eq,init}} + \Delta I/\sigma$ and $I_{\mathrm{eq,fin}} = Y_{\mathrm{eq,init}} + \Delta I$), giving

$$\dot{\boldsymbol{Y}} = \boldsymbol{A}\boldsymbol{Y} \quad \text{where } \boldsymbol{Y} = \begin{bmatrix} Y \\ I \end{bmatrix} \quad \text{and } \boldsymbol{A} = \begin{bmatrix} \frac{-\sigma}{P} & \frac{1}{P} \\ \frac{-\sigma\beta}{P\gamma} & \left(\frac{\beta}{P\gamma} - \frac{1}{\gamma}\right) \end{bmatrix} \tag{8}$$

In the case studied here, we take $\sigma = 0.356$, $P = 0.3$yr, $\gamma = 2.24$yr and $\beta = 1.42$yr. The resulting eigenvalues of $\boldsymbol{A}$ are a complex conjugate pair with positive real parts. The final equilibrium $\left[Y_{eq,fin}, I_{eq,fin}\right]^T$ is thus unstable, with nearby trajectories being oscillatory divergent.

The general solution - applicable in the unbounded phases of boom or bust - is

$$\boldsymbol{Y}(t) = Re\left[\boldsymbol{\Phi}\exp(\boldsymbol{\Lambda}(t - t_0))\boldsymbol{\Phi}^{-1}\boldsymbol{Y}(t_0)\right] \tag{9}$$

where $\boldsymbol{\Lambda}, \boldsymbol{\Phi}$ are the (complex) matrices of eigenvalues and eigenvectors of $\boldsymbol{A}$, and t_0 is the time at the start of the boom or bust.

At the top of the boom plateau, income is maximal and constant, and the second Phillips equation reduces to

$$\gamma\dot{I} = -I \text{ with solution } I(t) = I(t_0)e^{-(t-t_0)/\gamma} \tag{10}$$

The trickiest phase to model is the bottoming out of the bust phase. Obviously Investment is constant at its lower bound and Income Y, by virtue of the multiplier equation

$$\dot{Y} = -\frac{\sigma}{P}Y + \frac{1}{P}I_{min} \text{ with solution } Y(t) = \left(Y(t_0) - \frac{I_{min}}{\sigma}\right)e^{-\sigma(t-t_0)/P} + \frac{I_{min}}{\sigma} \tag{11}$$

continues to fall, at an exponentially-decreasing rate. The question is - how long does this continue? At what point does the boom pick up again? In order to answer this, the water level in the bucket needs to be tracked separately.

Although the bucket has bottomed out, the water level within it continues to fall, and this can be integrated until the point where the bucket is sufficiently empty that it will be lifted again from the floor by the spring. The second Phillips equation is not valid in this phase, because it was derived assuming a floating bucket.

Adopting a linear approximation to the orifice flow, the falling water level in the bucket is governed by

$$\dot{h} = -\frac{c}{A_b}(h - H) \tag{12}$$

Since the bucket is grounded we have $H = D = Y_{\mathrm{eq,fin}} + Y(t)$ (measuring heights in Machine units). Substituting $H(Y(t))$ into Eqn. (12) gives an exponentially-decaying forcing term:

$$\begin{aligned}
\dot{h} + \frac{c}{A_b}h &= \frac{c}{A_b}\left(Y(t_0)e^{-\sigma t/P} + Y_{\text{eq,fin}}\right) &(13)\\
\text{with solution }\ h(t) &= C_1 e^{-ct/A_b} + C_2 e^{-\sigma t/P} + C_3 &(14)\\
\text{where }\ C_2 &= \frac{c/A_b}{(c/A_b) - (\sigma/P)} Y(t_0) &(15)\\
C_3 &= \frac{c}{A_b} Y_{\text{eq,fin}} &(16)\\
\text{and }\ C_1 &= h(t_0) - C_2 - C_3 &(17)
\end{aligned}$$

This falling water level $h(t)$ can thus be tracked until force equilibrium is again returned to the bucket, whereupon the upward-pulling spring force can raise the weight of the excess water in the bucket, i.e. $h(t)$ falls until

$$\gamma_w A_b(h - H) = kz_0 \ \text{ i.e. when } \ h(t) = (Y_{\text{eq,fin}} + Y(t)) - \frac{L_2}{L_1}\phi I_{\min} \tag{18}$$

This criterion sets the time for which the bucket remains grounded, before the new boom phase begins.

All that remains is to prove that the Machine has settled onto a cycle. This is achieved by noting the uniqueness of the conditions at the "pinch-point" at the end of the boom plateau. Here Income $Y = Y_{\max}$ and Investment has decayed to the switch criterion $I = \sigma Y_{\max}$. Since the bucket is floating then vertical force equilibrium dictates that h is a function $h(Y, I)$ and is thus also uniquely determined at this switchpoint. At the end of any boom plateau, then, all three variables Y, I and h must take specific values.

Thus, for parameters as here where the final theoretical equilibrium is unstable, almost any initial transient will reach the upper-bound from where it will be guided to the unique pinch-point at the start of bust phase. There is no gradual convergence to a limit cycle such as might be observed in a smooth dynamical system: instead, the dynamics of the accelerator gather up any initial transients and funnel them directly onto the final limit cycle trajectory.

The solutions to the four separate phases of the cycle (together with the initial transient) are concatenated in the time domain in Fig. 5, with the experimental results from the Machine superimposed. One or two details are not perfectly captured, but given the approximations in the theory and the inaccuracies in the experimental data collection (particularly regarding Investment) the general level of agreement is surprisingly good.

The solutions are also presented on the (Y, I) phase plane and the $(Y, \dot{Y})$ phase plane in Fig. 6.

Economic Interpretation

The question as to whether the cyclic behaviour of the Machine has any relation to economic possibilities is a difficult one for an engineer. The Machine omits many economic realities (such as inflation) and numerous other factors were shut down for the simulations here (taxation, Government spending, interest rates, the foreign sector,

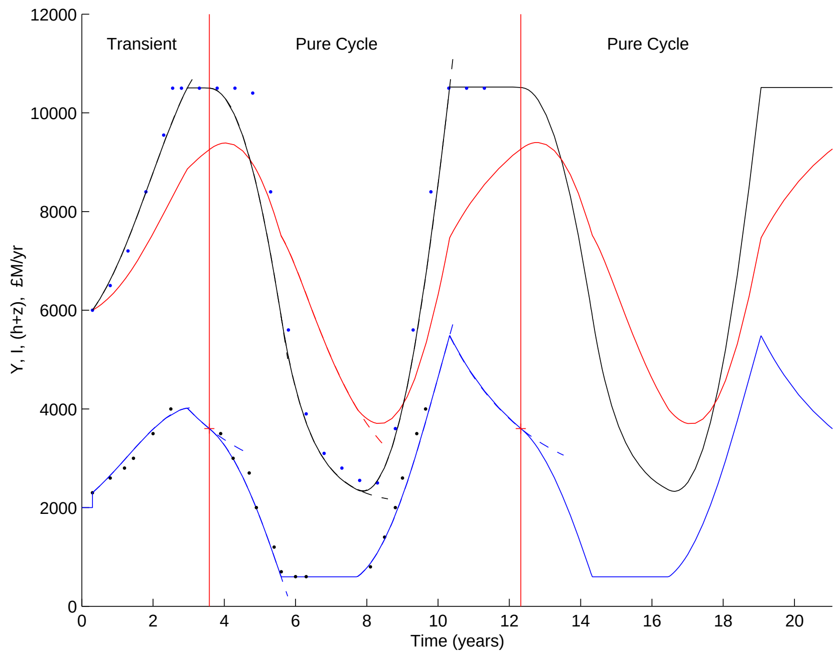


Figure 5: The solutions to the bounded accelerator-multiplier equations. The upper curve shows boom-bust cycles of income with a comparatively regular period of about 9 years, and the lower curve shows the corresponding investment behaviour. The water level in the accelerator bucket (in economic units) is also plotted as the intermediate curve. Physical results from the Machine simulation are shown as dots.

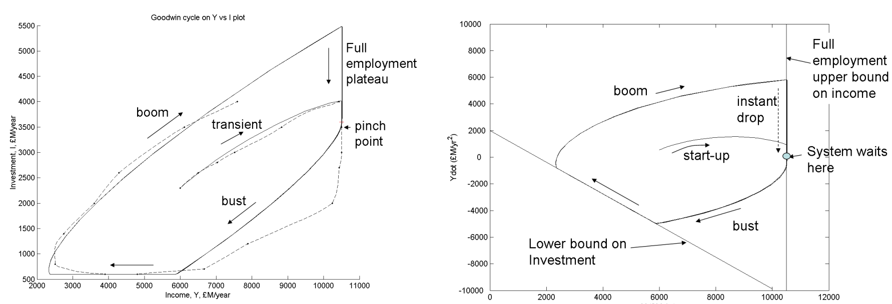


Figure 6: The limit-cycle on the (Y, I) plane (left) and on the $(Y, \dot{Y})$ plane (right);

etc.). During the rapidly rising phase, though, what remains is an analogue representation of the Keynesian multiplier-accelerator equations. Whether these equations have anything more than didactic value is beyond the scope of this paper, but without doubt, they form part of the discourse of economic history. More questionable here is the extent to which the current implementation provides an analogue representation of Hicks' and Goodwin's proposed model of the bounds.

For example, at the full-employment upper bound, Income Y has been limited by overspilling any excess (of main tank inflow over outflow) into the back of the Machine. The quantity of money in the main tank therefore does not rise during this phase, and it could be argued that perhaps the excess should be collected rather than removed[3]. The accumulated overflow monies could then be later reintroduced into circulation before the main tank level itself begins to fall at the start of the pure accelerator-multiplier phase of rapid decline. The main effect of such an adjustment would be to prolong the duration at the bound, and this in turn would allow more time for the accelerator to self-equilibrate thereby somewhat dampening its effect.

At the lower limit of zero net investment, it is more questionable as to what the Machine is actually simulating. It has been explained how during this phase, the bucket water level needs to be tracked separately. Inspection of the accelerator equations and their hydromechanical implementation reveals that the height h of water in the accelerator bucket is actually simulating an economic variable as yet unmentioned, namely the capital K. Specifically, the difference in water levels between that in the main tank and that in the accelerator bucket is a proportional measure of the excess (or deficit) of capital present relative to that which could maintain the existing income.

Assuming that the investment at the initial equilibrium is purely for replacement, and that the correct amount of capital is present to sustain the equilibrium income Y, then $K_{eq} = \beta Y_{eq}$ and $I_{replace} = rK_{eq}$ (where β is the (known) acceleration coefficient and r is the rate of depreciation) it follows that K and r can be determined at the initial equilibrium. The subsequent accumulation of capital can then be tracked by integrating

$$\dot{K} = -rK + I_{total} \tag{19}$$

The results of such an integration are shown in Fig. 7. Capital has been scaled by the acceleration coefficient, such that the income Y curve is the required (scaled) capital.

The cycle begins with capital accumulating during the initial rapidly-rising transient due to the increasing rates of investment. As income plateaus, capital is still lagging behind the required level, and so it continues to rise, the rate reducing as the two curves (actual and desired) approach each other. This reduction is the result of reduced rates of investment which, in turn, can no longer sustain the full income. The details at the precipice edge of the upper plateau depend upon whether the excess money has been lost or stored (as described previously), but whichever is the case, the system embarks on a rapidly-falling phase and a point is quickly reached beyond which there is an excess of capital (where the income and scaled capital curves cross). Net investment falls rapidly to zero leaving only depreciation. The conundrum now is: given that there is an excess of capital from here onwards, why should investment ever pick up again? Indeed, Goodwin [2], p77 stated

"When output has fallen, leaving general excess capacity, there is no reason to invest and the accelerator is dead: it can take 15, 50 or more years for the excess capacity to disappear, so that the cycle would be spending most of its time in depression."

[3]Conceptually, one can envisage equipping the main tank with a large impermeable floodplain where large volumes of water can be stored with negligible change of head and from where water can freely flow back to the main tank.

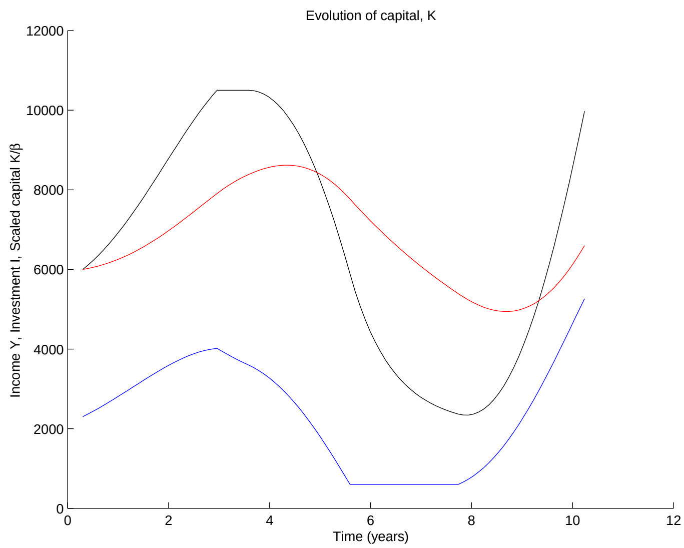


Figure 7: The middle curve shows the variation of (scaled) capital and its behaviour relative to Income (top curve) and Investment (lower curve).

The accelerator in the Phillips Machine, however, is not dead. Investment does pick up again, even though there is still an excess of capacity. The hydromechanical reasons for this were described earlier: the bucket water level continued to fall whilst the bucket was grounded until it was light enough for the spring to lift it up again. The economic analogue of this is more difficult to justify, but some insight can be found from an examination of the accelerator equation:

$$\gamma \dot{I} = \beta \dot{Y} - I \tag{20}$$

(where all variables are measured relative to the final theoretical equilibrium). In economic terms, this is the derivative of the equation for the evolution of capital

$$\gamma \dot{K} = \beta Y - K \tag{21}$$

with $\dot{K} = I$. Whilst the former equation admits the possibility that the right hand side can become positive (implying that I will pick up again) the latter equation suggests this is not realistic. Along the lower bound there is excess capacity, and there is no incentive to invest. The accelerator should therefore be disconnected here, in line with Goodwin's argument that it is now dead.

Sadly then, the cycles disappear. The rapid transient growth, the upper plateau and the subsequent collapse appear to be modelled with some reasonable degree of economic reality, but once the bottom is hit, the model should stay there: a more sophisticated model of the evolution of capital is required.

7 Summary and Conclusions

The Cambridge Phillips Machine was restored to full working order in 2003. It is now safe to operate, and has since been used in numerous enthusiastically-received public demonstrations. The basic accelerator-multiplier simulations that are usually demonstrated have been extended in the course of this paper in an attempt to encompass the occurrence of business cycles. Linear oscillations - growing, decaying, or (when there is a particular coincidence of parameter) persistent - could be readily performed by the Machine. If parameters were such that the system was oscillatory divergent, then the Machine could also embark on boom-bust limit cycles between the two nonlinear bounds of maximum income and minimum investment. However, closer inspection suggests that the model is not realistic at the bottom of the cycle, and that the accelerator equations break down there. Once the bottom is hit, there is excess capacity, the accelerator should be disconnected and a different model should be invoked to model just how long the economy will stay depressed.

In the simulations presented here, much of the Machine functionality has been shut down (interest rate variations, government policies on taxation, investment and borrowing, exchange rates and the foreign sector, etc.), and the effects of reintroducing these is a topic for future experiments. Such features could provide mechanisms for climbing out of the depression, and it is well-known that economists have differing views on how best to accomplish this. It is a subject of great topicality, and despite the initial indications, it would appear - sadly - that doing nothing is not a sensible policy - the economy will not climb out of depression all by itself.

8 Acknowledgements

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