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THE TIME-TO-BUILD TRADITION[♣] IN BUSINESS CYCLE MODELLING[♥]

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[♣] Prepared in homage to celebrate the eightieth anniversary of the publication of Tinbergen's path-breaking *Ein Schiffbauzyklus* and the sixtieth anniversary of the publication of Goodwin's *Nonlinear Accelerator and the Persistence of Business Cycles*.

[♥] The second author had the pleasure and privilege of direct and indirect instruction and years of inspiration on the matters dealth with in this paper by some of the pioneers of the relevant theory. In particular, of course, Richard Goodwin and Björn Thalberg, but also Trygve Haavelmo, Nicholas Kaldor and, especially, Jan Tinbergen (alas only very late in his noble life; see footnote 1, in the main text). Stefano Zambelli's influence, via innumerable discussions with the second author for over a quarter of a century, and through his important written works on Frisch and Kalecki, is pervasive. He is, however, not responsible for any remaining infelicities in this paper.

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Abstract

An important frontier of business cycle theorising is the 'time-to-build' tradition that lies at the heart of Real Business Cycle theory. Kydland and Prescott (1982) did not acknowledge the rich tradition of 'time-to-build' business cycle theorising - except in a passing, non-scholarly, non-specific, reference to Böhm-Bawerk's classic on Capital Theory (Böhm-Bawerk [1899]), which did not, in any case, address cycle theoretic issues. The notion of ‘time-to-build’ is intrinsic to any process oriented production theory which is incorporated in a macrodynamic model. We provide an overview of this tradition, focusing on some of the central business cycle classics, and suggest that the Neo-Austrian revival should be placed in this class of dynamic macroeconomics, albeit ‘traverse dynamics’ is itself to be considered as a fluctuating path from one equilibrium to another.

1 The Time-to-Build Tradition in Business Cycle Theory

> "That wine is not made in a day as long been recognized by economists (e.g., Böhm-Bawerk [1891]). But, neither are ships nor factories built in a day. A thesis of this essay is that the assumption of multiple-period construction is crucial for explaining aggregate fluctuations. ....
>
> Our approach integrates growth and business cycle theory. ….
> One very important modification to the standard growth model is that multiple periods are required to build new capital goods and only finished capital goods are part of the productive capital stock. Each stage of production requires a period and utilizes resources. Half-finished ships and factories are not part of the productive capital stock. "
>
> Kydland & Prescott (1982), p. 1345.

Apart from the gratuitous reference to Böhm-Bawerk's classic on capital theory, the oldest referenced paper in the Kydland & Prescott (henceforth, K & P) 'classic' is to the descriptive - questionnaire-based - article by Thomas Mayer (1960), which, in turn, refers only to work by that author, and none of them of any vintage earlier than 1953.

Thus, the whole noble business cycle theoretic tradition incorporating variations on the theme of 'time-to-build' - meaning by this not just the time length required to complete the building of plant to produce capital goods that can, in turn, be used in the production process, but also the lead and lag times involved between decisions to build, orders to be placed, delivery to be undertaken, and so on - all the way from Tinbergen's classic *Ein Schiffbauzyklus* (Tinbergen [1931])[1] to the Keynesian tradition of nonlinear business cycle

[1] Tinbergen's initiation into economics was partly due to the encouragement of his Physics mentor, the great Paul Ehrenfest, who advised his young assistant to contact Wicksell, in 1925, which he duly did on 23 June, 1925. Velupillai wrote Tinbergen on 3rd February, 1984, asking whether Tinbergen remembered this letter and, if so, for a copy of Wicksell's response, if there was one. Tinbergen's response to Velupillai, dated 10/2/1984, together with his letter to Wicksell, are both appended to this paper. Wicksell died on 3rd May, 1926. It is interesting to note, as pointed out by Hollestelle (2006; p. 790):

> "Ehrenfest's hand can also be seen in Tinbergen's famous analyses of the ship-

theories (Goodwin [1951], Strotz, et.al [1953]) is ignored. In between, there were the classics by Frisch (1933), Frisch & Holme (1935) and the series of pioneering contributions by Kalecki (1935, 1936, 1939), which also are ignored. These two traditions share a mathematical formalism in that the dynamic equation that these works reduce their rich macroeconomics to can be encapsulated in a canonical nonlinear difference-differential equation[2].

If we take the reference to Böhm-Bawerk in K & P, in the context of business cycle theory, seriously, then a reasonable expectation[3] would have been some mention of the rich, albeit controversial, tradition of Austrian Business Cycle theory linking 'Böhmian' capital theory, in the form of the *period of production*, with *industrial fluctuations*. The classic of this genre is, of course, Hayek's controversial little masterpiece, *Prices and Production* - which was, subject to searching criticisms, from many points of view, by Sraffa (1932), Hansen & Tout (1933), Hill (1933) and, above all, given the *provenance* of K & P, Frank Knight's series of critical essays on the *Austrian Theory of Capital* (Knight [1933], [1934]'

> building cycle, in which Tinbergen used the invention of his mentor, the adiabatic principle, to describe the periodic behaviour of the cycle."

The truth of this interesting observation can be verified in footnote 1, p.156, of the Tinbergen classic (italics added):

> "D.h. der Wert nimmt im Laufe der betrachteten Periode langsam ab. Streng genommen ist also eigentlich a auch eine Funktion von t. Wegen seiner langsam Veranderlichkeit ist es aber weitaus die einfachste Behandlungsweise, a voräfig als konstant zu betrachten und in der dann gefundenen Lösung als veränderlich einzusetzen. *Diese Methode ist der Physik bekannt als die Methode der adiabatischen Variabeln.* (*Vgl. P.Ehernfest,* Ann.d.Phys. Bd.51 [1926].S.327)."

[2] With characteristic perspicacity, referring to Tinbergen (op.cit), Schumpeter 'hit the nail on the head' (Schumpeter [1939], p.533; italics added):

> "This cycle [The cycle in Shipbuilding], made famous by Professor Tinbergen, serves to illustrate a lag phenomenon *incident to all time-consuming construction of plant and equipment* and therefore differs (also in other respects) materially from the hog case."

Schumpeter's 'time-consuming construction' is what K & P have 'dubbed' 'time-to-build'. A little bit of scholarship could prevent a great deal of square-wheel reinventions.

[3] Even a 'rational' one!

[1938])[4]. Indeed, this particular Hayekian theory comes closest to being an *Equilibrium Real Business Cycle Theory*, presaging and being a predecessor of modern RBC theory, but with at least two caveats: the first, is that the latter is not underpinned by a serious capital theory the way Hayek's attempted to be[5]; second, the former did not have - nor seek - the kind of theoretical technology that came to clothe modern RBC theory.

Then, there is the whole tradition of *replacement cycles*, initiated in Marx (1893)[6] and elegantly summarised in the language of linear algebra by Bródy (1970). It is this tradition that links up most coherently with the *traverse dynamics* and the general viability[7] of such paths of Amendola and Gaffard (1998). As a matter of fact, it is this *Marxian tradition* - we may refer to it this way, rather than as 'replacement cycles', which suggests only the purely technical aspects of durable goods replacements - that should be contrasted with the *Neo Austrian tradition* and, indeed, should be considered the foundation for the Amendola & Gaffard (op.cit) exercises[8].

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[4] Kaldor, whose intellectual adherence to Hayek's theory of capital and industrial fluctuations was most eloquently defended, especially against Frank Knight's penetrating criticisms, eventually turned against the Austrian visions, first in his brilliant criticism of Hayek's 'Concertina Effect' (Kaldor [1942]) and finally acknowledged his agreement with Knight at the famous 'Corfu Conference on Captial Theory' (Kaldor [1961], p.294):

> "Professor Haberler would know that he [Kaldor] had himself at one time defended Wicksell from the attacks of Professor Knight. He was now convinced that all he had written in defence of neo-classical theory was wrong and that Professor Knight was right."

[5] 'Attempted to be' is a serious qualification, given, in particular, Sraffa's devastating 'Wicksellian critique' (op.cit) of Hayek's claim that he was building on the foundations of Wicksell's reformulation of Böhm-Bawerk's capital theory.

[6] A concise, but characteristically erudite, summary of this tradition, linking it also the line of research begun by Tinbergen, is given in Schumpeter (op.cit, chapter IV, § E).

[7] By general viability is meant the *real*, *financial* and *human resource* (i.e., *labour*) feasibility of any traverse dynamics of a multisectoral dynamic economic system. The coherence between the two approaches can be gleaned from chapter 1.3 of Bródy (op.cit).

[8] After all, Böhm-Bawerk's *magnum opus* was itself devised and presented as an alternative to the Marxian system. The viability of any traverse dynamics in Marx's extended reproduction schemes took into account the durable good, labour and financial feasibility of any such path, whether 'steady'

Finally, there is the masterly work by Haavelmo, *A Study in the Theory of Investment* (Haavelmo [1960]), which may be referred to as a *non-stochastic* (ibid, chapter I, §3) macro*dynamic* foundation of investment theory, in complete contrast to the K & P approach. Haavelmo's '*Study*' is, in fact, a synthesis of all of the above mentioned approaches to encapsulate notions of 'time-to-build' in its manifestations of aggregate fluctuations. A particular application of Haavelmo's framework, in terms of the interaction of the 'time-to-build' and delivery time, placed in the context of a nonlinear Keynesian business cycle model of the Goodwin-type (op.cit), can be found in a remarkable - and much neglected - series of contributions by Björn Thalberg (1961, 1966).

The paper, which should be considered a small contribution to the doctrine-history of an aspect of business cycle theory, is organised as follows. In the next section a synopsis of the four pioneering business cycle theories where some notion of 'time-to-build' played a crucial role in determining the final mathematical form of the equation that underpinned fluctuations of one sort or another. Section 3 is an attempt to summarise the mathematical and economic lessons to be learned from 'time-to-build' modelling and an ingredient of business cycles. The concluding section is a brief methodological reflection on the lessons to be learned from 'time-to-build' modelling in what we call phenomenological macroeconomics. The addenda consist of two appendices. The first appendix showing some simulation results of a canonical ‘time-to-build’, nonlinear difference-differential equation model with increasing ‘precision’ incorporated, in the form of retaining higher order terms of the Taylor series. The classical nonlinear, endogenous, business cycle result of the genesis of a stable limit cycle, independent of initial conditions, is lost when the precision of the approximating equation is

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or not, whether initialised on a steady state path or not and, in any case, the notion of equilibrium was far richer than supply/demand consistency.

improved. The second appendix consists of Tinbergen's letter to Wicksell and Tinbergen's response to a query by Velupillai regarding the Tinbergen-Wicksell correspondence (see *appendix 2*).

2 The Canonical Difference-Differential Equations in Business Cycle Theory

> "The roots of the algebraic equation $\sum \alpha_\nu x^\nu = 0$ play a well-known role in the solution of the differential equation $\sum \alpha_\nu y^{(\nu)}(x) = 0$ .....
>
> Over a number of years a variety of economic and engineering problems .... has led to a study of difference-differential equations of which
>
> $$\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} a_{\mu\nu} y^{\nu}(x+\mu) = 0 \qquad (1)$$
>
> is a basic example. Here the algebraic equation is replaced by the transcendental equation
>
> $$\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} a_{\mu\nu} z^{\nu} e^{\mu z} = 0 \qquad (2)$$
>
> which has an infinity of roots. Sums of terms of the type $A_z e^{2z}$ over some or all of the roots of (2) (with grouping of terms if necessary to secure convergence) provide solutions of (1)."
>
> Wright (1961), p.136

The four canonical difference-differential equation models in macro dynamics are those that first appeared in Tinbergen (op.cit), Frisch (1933), Kalecki (1935) and Goodwin (1951). They are, discussed below:

- *Tinbergen (1931)*[9]

[9] Tinbergen, too, intellectually honest though he was, succumbed to the pointless temptation to add the well known caveat of all mathematical economics exercises (ibid, p.155):

$$f^{'}(t) = -af(t-\vartheta) \qquad (a > 0) \tag{3}$$

Where:

f (t) : total freight tonnage at time t (t : continuous, $t \in \Re$)

f' (t) : rate of change of freight tonnage (= ship building);

ϑ : a parameter, indicating the time period between decision to order extra tonnage and delivery of new ships ($\vartheta \in \Re$)

a : reaction coefficient, $a \in \Re$

Tinbergen's remarkable originality here was the behavioural assumption underpinning the accelerator dynamics encapsulated in (3): it was what later (in Goodwin [1951]) came to be called the 'flexible accelerator' (or the 'non-linear accelerator') with the equivalent of the difference between a 'normal' and 'actual' level of freight tonnage driving a positive feedback in the rate of ship building.

- *Frisch (1933)*

$$\ddot{x}(t) = \left(\frac{s\mu}{\varepsilon} - \lambda r\right)\dot{x}(t) + \left(\frac{s\mu}{\varepsilon}\right)\dot{x}(t-\varepsilon) + \frac{sm}{\varepsilon}[x(t) - x(t-\varepsilon)] \tag{4}$$

Where:

x : 'yearly production of consumer's goods';

m : 'the *total* depreciation on the capital stock associated with the production of a unit of consumer's goods', $m \in \Re^{+}$;

"Schlie[ss]lich mu[ss] die Lösung noch der Bedingung genügen da[ss] sie überhaupt einen ökonomischen Sinn hat: sie soli also z.B. reell und endlich sein."

μ : 'the size of the capital stock that is needed directly and indirectly in order to produce one unit of consumption per year', $\mu \in \Re^{+}$;

ε : 'technically given constant' - essentially the 'time-to-build' parameter, $\varepsilon \in \Re^{+}$;

S : the *encaisse désirée* parameter for the production of capital goods, $s \in \Re^{+}$.

This is, of course a linear difference-differential equation - but the economic and mathematical reasons given for its genesis, in Frisch (1933), are untenable. Frisch begins by assuming the equivalent of a non-linear 'flexible accelerator' relationship between the production of consumption goods and the *encaisse désirée*,[10] but assumes, 'as a first approximation the relationship to be linear', and works with:

$$\dot{x} = c - \lambda\omega \tag{5}$$

Where:

$c, \lambda \in \Re^{+}$

Had Frisch removed the 'first approximation' of a 'linear relationship', the resulting dynamics in the production of consumption goods can be shown to have the form (Velupillai [1992], p.64, equation (10)):

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[10] It may well be worth quoting Frisch in detail on this point simply because it has, to the best of our knowledge, never been made explicit (ibid, pp. 179-180; italics in the original):

> "In the boom period when consumption has reached a high level, … , consumption is one of the elastic factors in the situation. It is likely that this factor is one that will yield first to the cash pressure. To begin with this will only be expresses by the fact that the rate of increase of consumption is slackened. Later, consumption may perhaps actually decline. Whatever this final development it seems plausible to assume that the *encaisse désirée* ω will enter into the picture as an important factor which, when increasing, will, after a certain point, tend *to diminish the rate of increase* of consumption."

$$[1 - s\mu g(x,\dot{x})]\ddot{x} - (r + sm)g(x,\dot{x})\dot{x} = 0 \tag{6}$$

Where:

$$f'(.) = g(x,\dot{x})$$

It can be seen, by a simple inspection of (6), that even a 'linear' approximation of $g(x,\dot{x})$ results in a second-order, non-linear, differential equation. This equation is capable of generating the kind of oscillation Frisch thought should be the object of study, for the interaction between theory and observation. But ignoring the natural strategy of removing the 'first approximation', Frisch claims that the dynamic equation for the production of consumption goods that was generated by sticking to the 'first approximation' of linearity - the linear, nonflexible, accelerator - 'is too simple to give rise to oscillations' (Frisch, op.cit, p. 180), and he goes on:

> "The system considered above [i.e., with the 'first approximation' of linearity] is thus too simple to be able to explain developments which we know from observation of the economic world. There are several directions in which one could try to generalize the set-up so as to introduce a possibility of producing oscillations."

After mentioning the possibility of taking the routes suggested by Keynes, Fisher and Marx (essentially, Kalecki), he opts for what he calls 'Aftalion's point of view with regard to production' (ibid, p. 181):

> "The essence of this consists in making a distinction between the quantity of capital goods whose production is started and the activity needed in order to *carry to completion* the production of those capital goods whose production was started at an earlier moment. The essential characteristics of the situation that thus arises are that the activity at a given moment does not depend on the decisions taken at that moment, but on decisions

> taken at earlier moments. By this we introduce a new element of discrepancy in the economic life that may provoke cyclical oscillations."

Thus enters the untenable reason for the introduction of the 'time-to-build' assumption - realistic though it may be - in a macro dynamic theory, albeit in a non-representative agent, non-optimum, yet entirely deterministic macro dynamic context.

In this sense of sticking to an untenable - both from economic and mathematical points of view - 'approximation', Frisch's later criticism of Kalecki seems highly questionable (Frisch & Holme [1935], p.225):

> "The imposition of the condition [by Kalecki (1935)] that the solution shall be undamped is in my[11] opinion not well founded. It is more correct, I think, to be prepared to accept any damping which the empirically determined constants will entail, and then explain the maintenance of the swings by erratic shocks. This would be an explanation along the lines indicated in my paper in the Cassel volume."

Moreover, even this methodological point by Frisch (& Holme) - that 'it is more correct'[12] to 'explain the maintenance of the swings by erratic shocks' reiterated as a dogmatic credo for mathematical modelling of business cycles at the frontiers of macrodynamics, based on the so-called substantiation in Frisch (1933), has been shown to be vacuous by Zambelli's fundamental result that the famous 'Rocking Horse' does not rock (Zambelli [2007])[13].

- *Kalecki (1935)*

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[11] It may well be a simple 'slip of the tongue' that 'my' is used - but not just once and not only 'my', but also 'I' - in a *joint* paper! The 'my' obviously refers to Frisch (and not Holme).

[12] What is the epistemological status of an assertion like 'more correct' in this context? It is a pity that Frisch's distinguished colleague, Trygve Haavelmo, debunked the methodological adherence of substantiating a theory on the basis of consistency with observations (Haavelmo [1940]), only half a decade *later*.

[13] Indeed, the infelicities in Frisch's highly celebrated Cassel Festschrift paper extends even to an important mis-attribution even of Wicksell's original reference to the 'Rocking Horse' (ef. Velupillai, op.cit., footnote 4, p. 70).

$$\dot{I}(t)=\frac{m}{\theta}[I(t)-I(t-\theta)]-n[I(t-\theta)-U] \tag{7}$$

Which, by writing:

$$J(t)\equiv I(t)-U \tag{8}$$

Can be represented more simply as:

$$\dot{J}(t)=\frac{m}{\theta}[J(t)-J(t-\theta)]-nJ(t-\theta) \tag{9}$$

Where:

$I\ (t)$: Investment orders at time t, ($\in \Re$);

U : (constant) depreciation factor ($\in \Re^{+}$);

θ : the average gestation lag, for the economy as a whole, between decisions to invest (order) and delivers of final (capital) goods;

$m, n\ (\in \Re)$: linearization parameters of the non-linear Investment function (see, equation (10, p. 331, *ibid*):

$$\frac{I(t)}{K(t)}=\phi\left(\frac{C_1+A}{K}\right) \tag{10}$$

Where:

$K(t)$: capital stock at time t;

A : gross accumulation equal to the production of capital goods;

C_1 : constant part of the consumption of capitalists;

Several comments should be added to the general tendency to refer to (7) (or, more frequently to (9)) as 'Kalecki's model of the cycle' which is, 'from a mathematical point of view … a differential equation with a delay parameter' (Szydlowski [2002], p.698). The main economic point is that there is no mathematical reason, underpinned by any compelling economic reason, for 'Kalecki's model of the economic cycle' to be anything other than a straightforward high-order difference equation. Secondly, there is no justification for the linearization mentioned above. Thirdly, the non linearized Kalecki model of the business cycle would be given by (see Velupillai [1997], equations (21) & (22), p. 261:

$$\frac{K(t)-K(t-1)}{K(t-\theta)} = \phi\left\{\frac{C_1+U+\frac{1}{\theta}[K(t)-K(t-\theta)]}{K(t-\theta)}\right\}-U \tag{11}$$

This is a non-linear difference equation and, paradoxically, even if ϕ is now linearized, the final equation will remain a *non-linear difference equation*! Sixty years ago, in his masterly review of one of the great modern classics of endogenous macroeconomic cycle theory, Hicks (1950), Richard Goodwin reflected on such equations with characteristic prescience (Goodwin [1950], p.319, footnote 6):

> "Combining the difficulties of difference equations with those of non-linear theory, we get an animal of a ferocious character and it is wise not to place too much confidence in our conclusions as to behavior."

To substantiate Goodwin's prescience on being careful 'not to place too much confidence in our conclusions as to behavior', we could add the following conjecture[14] :

[14] This is a comprehensively non-rigorous 'conjecture', as stated. However, it is quite easy to state this more rigorously by showing, first, the equivalence - in some formal sense between this dynamical system and a suitably initialised Turing Machine and, then, invoking the theorem of the *Halting*

Conjecture 1 *For any economically interesting nonlinear function ϕ, the attractors of (11) are algorithmically undecidable.*

The only reason why 'Kalecki's model of the economic cycle' is 'from a mathematical point of view ... a differential equation with a delay parameter', i.e., a difference-differential equation, is that Kalecki chose to sum the total of orders allocated during a period $(t-\theta, t)$ *continuously* - for which he could not have had any kind of economic data - rather than in *discrete time*. Had he chosen the latter part, the result would have been (11), above.

• *Goodwin (1951)*

$$\varepsilon\dot{y}(t+\theta)+(1-\alpha)y(t+\theta)=O_A(t+\theta)+\varphi[\dot{y}(t)] \tag{12}$$

Where:

y : aggregate income;

θ : one half the construction time of new equipment;

$\varphi(\dot{y})$: the flexible accelerator;

O_A : the sum of autonomous outlays ($\beta(t)$ and *l(t)*);

A more direct way to look at this would be to write it out as:

Problem for Turing Machines. It must be emphasised that we are not under the sanguine impression that any discrete dynamical system implies, and is implied by, an algorithm in its formal, recursion theoretic or constructive sense. However, respecting the natural data types in economics would entail that the discrete dynamical system (11) can be considered a natural algorithm, but, of course, subject to the classic theorem of the *Halting Problem for Turing Machines*.

$$\varepsilon\dot{y}(t+\theta)+(1-\alpha)y(t+\theta)=\varphi[\dot{y}(t)] \tag{13}$$

Where, now, $O_A(t+\theta)$ is assumed to be a constant and *y(t)* is redefined as a deviation from its unstable, repelling, equilibrium value, $\dfrac{[\beta(t)+l(t)]}{(1-\alpha)}$ and time units are shifted by θ. This equation is a *non-linear difference-differential equation*, derived with impeccable macroeconomic logic. Unfortunately, Goodwin decided to approximate this by 'expanding the two leading terms in a Taylor series and dropping all but the first two terms' (ibid, p.12), to derive the famous (unforced) Rayleigh-van der Pol non-linear differential equation:

$$\varepsilon\theta\,\ddot{y}+[\varepsilon+(1-\alpha)\theta]\,\dot{y}-\varphi(\dot{y})+(1-\alpha)y=0 \tag{14}$$

Fortunately, however, in an early electro-analogue (as distinct from an analytical or digital) study of (12), Strotz, et.al., (1953), found a multiplicity of limit cycles and a breakdown of the notion of 'independence of initial conditions' of such cycles, and reached the interesting and important conclusion[15] that (pp. 406- 7; italics added):

> "The multiplicity of cycles that has been observed [in the analogue simulations] *can be ascribed to the presence of the difference term*. Had Goodwin approximated his nonlinear difference-differential equation by using the first four terms of the Taylor series expansion of ['the two leading terms'], the resulting approximating equation would have bee a nonlinear differential equation of the fourth order, which we believe would have had two limit cycles solutions rather than one, both dependent on initial conditions. Improving the approximation by retaining more terms of the Taylor's expansion would increase the order of the differential equation and this would increase the number of

[15] As a part of illustrating the multiplicity of solutions and the dependence on initial conditions of Goodwin's model, we have replicated Strotz, et.al., (1953) simulations and analysed the model for higher orders using a digital computer. The results show that as the order of the differential equation increases the system tends to have multiple solutions, all depending upon the initial conditions, thus emphasizing the need for further investigation. (See *appendix 1*)

> solutions provided by the approximation. To the extent that this is generally true of nonlinear mixed systems, economic theory encounters a methodological dilemma. .... If mixed systems seem to be required, this implies that we must in general expect a multiplicity of solutions. The resulting indeterminacy must then be overcome by specifying the initial conditions of the model."

This conclusion is the most important one in the whole tradition of 'time-to-build' modelling, in the context of business cycle theory. It identifies and demonstrates almost exactly the nature of the role played by 'time-to-build' assumptions, within the context of a macro dynamic theory, in generating endogenous cycles and - instead of independence of initial conditions - shows independence of *ad hoc shockeries*[16], or, exogenous shocks. Unlike in the case of the other three pioneering formal contributions to this tradition, considered above, in this approach very few - if any - arbitrary, ad hoc, approximations, without economic rationale, were made in deriving the final form of the dynamic equation in the considered variable.

3 Mathematical and Economic Considerations in Solving Non-Linear Difference-Differential Equations

> " *Thus from the standpoint of stability of self-excited oscillations, a linear d. d. e* [difference-differential equations] *is unable to account for the observed facts*, just as it was impossible to account for the existence of self-sustained oscillations on the basis of an ordinary linear d.e [differential equation] . ....
>
> Hence, if one tries to fit the oscillations appearing in retarded systems into the framework of the linear theory of d.d.e., one has exactly the same difficulty that was experienced in the theory of ordinary d.e. when one tried to fit self-sustained oscillations into a similar linear process.
>
> Obviously, the only issue from this situation is to investigate the

[16] In the felicitous phrase coined by Richard Day to describe Lucasian business cycle theory.

non-linear d.d.e. In fact, all observed oscillations ... start spontaneously from rest as soon as a certain threshold value of a parameter is reached; moreover, they generally exist not only for one isolated value of the parameter (as indicated by the linear theory), *but for a certain interval* of these parameter values; finally, oscillation persists with a definite stationary amplitude for a given value of parameter."

Minorsky (1962 [1974]), pp.52-3; italics in the original

Kalecki's equation, (9), can be rewritten in the form:

$$\dot{J}(t)-\frac{m}{\theta}J(t)+\left(\frac{m}{\theta}+n\right)[J(t-\theta)+J(t-\theta)]=0 \tag{15}$$

In more general notation, this can be written as:

$$\dot{x}-ax+bx_{\theta}=0 \tag{16}$$

Where:

$x_{\theta} \equiv x(t-\theta)$ (or, depending on the context, $x_{\theta} \equiv x(t+\theta)$)

Had Kalecki done what Goodwin did, then, first shifting the time coordinate by θ units, (16) can be rewritten as:

$$\dot{x}_{\theta}-ax_{\theta}+bx=0 \tag{17}$$

Then, a Taylor series expansion of the leading term gives:

$$\dot{x}_{\theta}=\dot{x}(t+\theta)=\dot{x}(t)+\frac{\theta}{1!}\ddot{x}(t)+\frac{\theta^{2}}{2!}\dddot{x}(t)+...... \tag{18}$$

Then, 'approximating' the linear difference-differential Kalecki equation by an 'equivalent' purely differential equation will retain the linear form and the above strictures of Minorsky can be shown very easily to be substantiated. Thus, Frisch was correct in his criticism of Kalecki, only because the latter's aim was to generate an endogenous cycle theory. Had Kalecki not approximated his ϕ (in equation (10), above), the Frisch critique would have been as inapplicable as it was to the final form of the Goodwin equation (12) which was, partly, devised to counter the Frischian, 'exogenous', *ad hoc shockeries*, methodology in mathematical business cycle theorising. From equation (18) the meaning of what Strotz et.al. (op.cit) did can also be gleaned.

Conversely, had Goodwin worked only with the linear accelerator, the general form of his differential-difference equation, in the above notation, would have been

$$\ddot{x} + px + \lambda x_{\theta} = 0 \tag{19}$$

The characteristic equation of which would be:

$$f(z) = z^2 + pz + \lambda e^{-\theta z} \tag{20}$$

Substituting $z = \alpha + i\omega$ in (20), separating the real and imaginary parts respectively and considering only harmonic values (i.e., $\alpha = 0$), we get:

$$\cos\beta_1 = \frac{\omega_1^2}{\lambda_1} \text{ and } \sin\beta_1 = \frac{\rho\omega_1}{\lambda_1} \tag{21}$$

$$\text{co}\tan\,\beta_1 = \frac{\omega_1}{p} = \left(\frac{1}{p\theta}\right)\beta_1 \tag{22}$$

and,

$$\beta^4 + \theta^2 p^2 \beta^2 - \lambda^2 \theta^4 = 0 \tag{23}$$

Analysing these equations give the basis for the Minorsky strictures in the opening quote for the following reason. There are two sets of roots: one set, $\beta', \beta'', ...$, independent of the variable parameter λ, can be called the set of *fixed roots*; the second set is given by the positive root of (20):

$$\beta_{11}(\lambda) = +\sqrt[2]{-\frac{p^2\theta^2}{2} + \sqrt[2]{\frac{p^4\theta^4}{4} + \lambda^2\theta^4}} \tag{24}$$

Thus, $\beta_{11}(\lambda)$ is a monotonically increasing function of λ, with $\beta_{11}(0) = 0$. Hence, as λ increases continuously, from $\lambda = 0$, $\beta_{11}(\lambda)$also moves continuously and for some value, say λ_1 could coincide with one of the above mentioned *fixed roots*, say β'; i.e.,

$$\beta' = \beta_{11}(\lambda) \tag{25}$$

and so on for, respectively[17], β''' and λ''', β''''' and λ''''', and so on. At these equalities, (19) and (20) have a common harmonic root:

$$\beta_1 = \theta\omega_1 \tag{26}$$

and:

$$f(i\omega_1) = 0 \tag{27}$$

As λ continues to increase, to a discrete sequence of values of λ, say, $\lambda_1^{'}$, $\lambda_1^{''}$, ... there will correspond also a discrete sequence of harmonic frequencies, say, ω_1, ω_1', ... The

[17] In view of (16) only the first, third, ... , fixed roots are relevant.

key result is that *the only point at which the dynamics can remain in a stationary state is precisely when λ is equal to a harmonic value*.

This is the thrust of Frisch's objection to Kalecki and Goodwin's indictment against linear theory and the meaning of the opening strictures against linear theory by Minorsky. This is also the kind of analysis that can make sense of Frisch's epistemological phrase on a 'more correct' theory. Essentially, this parallels the idea that structurally unstable dynamical systems - such as the Lotka-Volterra equations - should not be harnessed for modelling naturally occurring dynamical systems since they are highly unlikely to be meaningfully observable. Finally, this is also the way meaning can be attached to the results of Strotz et.al (op.cit).

Three concluding observations may be made.

What of the general non-linear difference-differential equation theory and why have economists shunned modelling in this framework? Almost seventy years of deep research on the general theory of non-linear difference-differential equation studies, from Wright (1946) and Brownell (1950, by way of the classic textbook of Bellman & Cooke (1963) and the monograph of Mohammed (1978), to Hale (1993) and beyond, has gone unheeded in macrodynamics. Why? We have no coherent answer to this simple - even simplistic - question. The natural mathematical dynamic framework for modelling 'time-to-build' processes in the context of business cycle theory appears to be the general non-linear difference-differential equation[18]. This is what we tried to show in the discussion of the

[18] Even more compellingly, given the nature of economic data types - that economic variables and parameters can, at best, only be rational valued - it must be the obvious way to model any dynamic process in economics. Such a formalization could easily be encapsulated within the general scheme of Diophantine Dynamics, a branch of Computable Economics.

'Kalecki model of the economic cycle', above.

Secondly, where does this leave the kind of linear dynamical systems that underpin the 'time-to-build' tradition emanating from the Kydland & Prescott research program? Are they not subject to the 'Minorsky strictures'? Indeed, they are - and to even more analytical strictures because the K & P tradition also claims computability. But developing these strictures has to be left for a different exercise.

Thirdly, what of the Neo Austrian 'traverse dynamics', as an example of 'time-to-build' dynamics as a 'disequilibrium process'? Before responding to this rhetorical query, it may be useful to recall yet another of Tinbergen's important reflections on an issue that is of relevance here. In Tinbergen (1943, p. 45)[19]:

> “[T]he theory of the business cycle contains the certain controversies derive from the part attributed to positions of equilibrium in the explanation of the business cycle; there are here two different and contrasting views: (a) the business cycle represents a movement around an equilibrium; (b) it is a movement between between two equlibriae[20]. The first view is expressed in many econometric models (Note: I may refer to Kalecky's[21] and my own work; but there are many other examples.)"

Thus, it is clear that all four pioneering theories considered in the previous section belong to the first of the two classifications suggested by Tinbergen[22]. It is our view that the Neo Austrian approaches should be considered in the second class.

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[19] The original is in Danish, but Velupillai was given, by Richard Goodwin, the original typescript of an English translation prepared by Tinbergen for Goodwin. The quote here is from this typescript. Incidentally, Goodwin himself considered this characteristically elegant paper by Tinbergen one of the pioneering contributions to the nonlinear theory of the business cycle (op.cit, p.2, footnote 3).

[20] This is the spelling in Tinbergen's translation.

[21] This is the spelling in Tinbergen's translation.

[22] Obviously, New Classical dynamics - whether of business cycles or anything else – belong also to the first of Tinbergen’s two categories.

Now to the third, rhetorical, query. Amendola and Gaffard (op.cit, p.25) note that:

> "[I]n the analysis of [Hicksian] Traverse, ... the adoption of a superior technique [is considered] as a process taking place sequentially over time. The explicit consideration of the time structure of the production process and if its intertemporal complementarity (with a focus on *the phase of construction* of a 'new' productive capacity and on its *coming necessarily before the phase of utilization*) allows to illuminate the fact that a change of the technique in use necessarily implies a change in the age structure of productive capacity and hence a dissociation of inputs from output and of costs from proceeds. We are in fact here clearly in an 'out-of-equilibrium' context … ."

This kind of 'traverse dynamics' is a path from one growth equilibrium - or one steady state growth path - to another. In the 'time-to-build' tradition that is tied to endogenous business cycle theory, on the other hand, 'traverse dynamics' is not a 'disequilibrium' thread linking two equilibrium configurations. As a matter of fact, we subscribe to the view that this particular 'traverse dynamics' vision, clearly and candidly expressed and described by Amendola and Gaffard, is an incoherent vision; our stand is substantiated by rigorous demonstration by Gunnar Myrdal against the Lindahlian concept of periods of temporary equilibria linked by points or time at which varieties of instantaneous changes occur (Myrdal [1931]). It is not surprising that this is a vision 'resurrected' by Hicks, who did more than anyone else to work within the Lindahlian framework of temporary equilbria, separated by periods during which disequilibria can emerge.

Unless the Neo Austrian notion of 'traverse dynamics' is placed within the context of business cycle theory, where 'traverse' is 'time-to-build' and is intrinsic to the dynamics of the system, it will remain, at best, a pseudo-dynamic process with provable indeterminacies. In particular, it is easy to prove that 'traverse dynamics', when formalised effectively, is undecidable in the precise sense of computable economics.

But this is a wholly additional consideration, beyond the scope of the narrow focus on the 'time-to-build' tradition in business cycle theory, that was the theme of this paper.

4 Brief Concluding Methodological Reflections

> "I must not be too imperialistic in making claims for the applicability of maximum principles in theoretical economics. There are plenty of areas in which they simply do not apply. Take for example my early paper dealing with the interaction of the accelerator and the multiplier. This is an important topic in macroeconomic analysis.
> ….
> My point in bringing up the *accelerator-multiplier* here is that it *provides a typical example of a dynamic system that can in no useful sense be related to maximum problems*. By examining the sick we learn something about those who are well; and by examining those who are well we may also learn something about the sick. The fact that the accelerator-multiplier cannot be related to maximizing takes its toll in terms of the intractability of the analysis.
> Samuelson, 1970 [1972], pp. 12-13; italics added.

In two methodological senses RBC modelling, incorporating the 'time-to-build' assumption, is perfectly coherent: in basing its foundations in optimization and in interpreting observed behaviour as optimum – rational - reactions to exogenous disturbances to an equilibrium configuration. Thus, also, belonging to the first of Tinbergen's above two classificatory characterizations, but with the added proviso that even when 'out-of-equilibrium' behaviour is rational. Obviously, the same cannot be claimed by any of the four theories of the business cycle discussed in section 2. They may be described as being fluctuations in aggregate variables in *phenomenological* macroeconomics[23], where not maximization but 'conservation' principles are invoked. In terms of concepts used in RBC modelling, this refers to what is called 'calibration' in that research tradition. Calibrating, for example, the parameters of an aggregate production function of the Cobb-Douglas type would be equivalent to generating conservative cycles in phenomenological macroeconomics. This is the kind of assumption that leads to 'relaxation oscillations' in non-linear models of the business cycle in phenomenological macroeconomics. Failure of this kind of conservation principle - for example in linear dynamic models - leads to unstable, non-cyclical, dynamics (as pointed out by Frisch's critique of Kalecki's model).

Tinbergen's two-fold characterization of macroeconomic dynamics may not be exhaustive. Tinbergen, in common with all analytical economists who came before and after him, characterized the interpretation of aggregate fluctuations on the basis of one or the other of equilibrium norms: either the observed fluctuations are a deviation from an equilibrium (or equilibria); or, they are movements between equilibria. But is it really true that these are the only ways to characterise any observable aggregate dynamics? Surely, it is also possible that observed fluctuations are independent of any equilibrium norm? In other words, is it possible to construct (observable) dynamical systems that can be studied without any equilibrium norm? We conjecture that non-maximum dynamical systems - i.e., dynamical systems 'that can in no useful sense be related to maximum problems' - are those that display intrinsic dynamics, without any anchoring in any kind of equilibria. A constructible example of such dynamical systems are those that are capable of *'computation universality'* (cf., Velupillai [2011]).

[23] A phrase we have coined on the basis of the hint from Tinbergen's use of Ehrenfest's adiabatic principle in the formalization and analysis of the *Schiffbauzyklus*.

We have referred to Myrdal's critique of Lindahl's temporary equilibrium dynamics[24] as an example of an incoherence in Neo Austrian 'traverse dynamics'. This is part of a more formal criticism of any kind of interperiod (dis)equilibrium dynamics linked by alternative equilibria, as in Lindahl-Hicks or in Hicksian Neo Austrian 'traverse dynamics'. In terms of formal dynamical systems theory the critique is about the dynamics at a boundary separating basins of attraction. Now, dynamical systems capable of computation universality reside only at the boundaries of basins of attraction. All formal macroeconomic dynamic models are constrained to lie within one or another of the basin of attraction of a given dynamical system and, therefore, eventually analysable in terms of equilibrium norms. This is not the case for dynamical systems that reside on the boundaries of basins of attraction - i.e., dynamical systems capable of computation universality[25].

The rich tapestry of dynamics implied by incorporating interesting 'time-to-build' assumptions in macroeconomics, particularly in its phenomenological versions, could give rise to wholly new, non-equilibrium, non-maximum, research paradigms. The lessons to be learned from the classics are inexhaustible and, at least in this sense, the role of the history of economic thought should not be underestimated.

[24] Unfortunately, this critique appears only in the Swedish version (Myrdal, op.cit, pp. 227-230) of *Monetary Equilibrium*, and was removed from both the German and English translations.

[25] A part of what we have in mind is discussed cogently in Pincock (2009) in terms of boundary layer dynamics of the Navier-Stokes equation.

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Appendix 1

The aim of this appendix is to study more 'precise' approximations to Goodwin's 'time-to-build' underpinned nonlinear accelerator model and investigate its dynamics through simulations. One of the very insightful, simulational, studies on Goodwin's model can be found in the work of Strotz, et.al., (1953). Using an electro-analog computer, Strotz, et.al., analysed the *formal properties* of Goodwin's model (see equation A.1).

$$\varepsilon y'(t+\theta)+(1-\alpha)y(t+\theta)=O_A(t+\theta)+\phi[y'(t)] \qquad \text{(A.1)}$$

Where:

y : aggregate income;

θ : one half the construction time of new equipment;

$\phi(y')$: the flexible accelerator;

O_A : the sum of autonomous outlays ($\beta(t)$ and *l(t)*);

They did so by simulating the nonlinear *difference-differential* system for various initials conditions and *higher order approximations of the Taylor series approximation of the above canonical equation*. The results showed that Goodwin's model is *sensitive* to 'initial conditions'[26] and there are "at least twenty-five other limit cycles that are also solutions to the same equation, indicating that there are an infinite number of additional solutions." (ibid, p: 398) Moreover, as Strotz, et.al., pointed out, if Goodwin had *not* approximated his non-linear *difference-differential* model by a second order *differential* equation (A.2), by taking only the first two terms but with more terms of the Taylor series expansion of the two leading terms, the model would have exhibited a *richer* dynamics.

$$\varepsilon\theta y''(t)+\left[\varepsilon+(1-\alpha)\theta\right]y'(t)-\phi[y'(t)]+(1-\alpha)y(t)=0 \qquad \text{(A.2)}$$

[26] We are not referring to the well known property of nonlinear dynamical systems known as 'sensitive dependence to initial conditions' (SSIC). Here, we simply mean that the reduction of (A.1) to the Rayleigh-van der Pol type nonlinear differential equation shown the existence of limit cycles independent of initial conditions. Taking better approximations to A.1 shows that this independence breaks down.

In fact, “[i]mproving the approximation by retaining more terms of the Taylor's expansion would increase the order of the differential equation and this would increase the number of solutions provided by the approximation” (ibid, p: 407), all depending upon the initial conditions.

This result emphasized, and continues to emphasise the need for further studies of a simulational kind, to learn how to approach the analytical solutions and properties of its attractors sand their dependence on the structure of sets of initial conditions. This appendix is structured such that, first, we replicate Strotz, et.al, results, by using a digital computer[27], and then go beyond *order 4* to investigate how the system would behave and evolve over time.

Simulations:

In 1953, Strotz, et.al., simulated Goodwin’s model, by using an electro-analog computer, and found the model to have multiple solutions depending on the initial conditions. They did this by systemically altering the initial parameter values and orders of the system to see if the set of solutions changed or not. As the computers in 1950s were in a developing stage they could not analyse for a wide range of values and there was “an *error* in the quantitative analysis of the circuit” (ibid, 398). Therefore in this exercise, we have replicated the simulation, for the parameters (see Table 1), in Strotz, et.al., paper by using a digital computer (see figure 1 *a, b, c* – 5 *a, b, c*). The results obtained in our analysis showed a great deal of similarities with the results obtained by Strotz, et.al., but with some minor differences, too. This might be due to the difference in structure of the input equation that is being fed in the computer and also because of the processing limitation of the computer itself. For example, Strotz, et.al, use equation (A.1) to build an electrical circuit to simulate the nonlinear *difference-differential* equation, whereas we take the Taylor series expansion of the two leading terms (i.e., $y'(t+\theta)$ and $y(t+\theta)$) and approximate the nonlinear *difference-differential* equation to a nonlinear *differential* equation and then simulate the system.

In fact, the higher modes used in Strotz, et.al., are given according to the orders of oscillations (in terms of frequency) while in our system the orders increase

[27] All the simulations have been carried out using Matlab. The code used for simulating the differential equation is *ode45,* which uses 4th and 5th order Runge-Kutta formulas.

according to the number of terms retained in the Taylor series expansion of the two leading terms. Moreover, as Strotz, et.al., write (ibid, p: 398), “[t]here are several characteristics of the higher-mode solutions which are peculiar to the apparatus [i.e., Electro-Analog computer] used and which introduce an error in the quantitative analysis of the circuit.” Because of the limitation of their apparatus, they did not have any provisions to control the initial values of $y(t)$, $\dot{y}(t)$, and $\varphi[\dot{y}(t-\theta)]$ at t=0; therefore, whenever the system is operated, the solution took a form of some higher modes and so they only analysed the modes for which the results are replicable. For example, the solutions obtained for the systems of mode higher than 4 were *unstable*, and due to the problem of replication of results, they confined their analysis to modes up to 4 (ibid, p:401). The discrepancy in the results might be due the above limitations but unlike *their* electro-analog computer, we were able to go beyond order 4.

As an example, we have illustrated the cases in which the nonlinear *difference-differential* equation is approximated by retaining 3, 4 and 5 terms in the Taylor’s expansion of the two leading terms and the nonlinear *differential* equations thus obtained are of order 3, 4 and 5 respectively (see A.3, A.4, and A.5).

$$\varepsilon\frac{\theta^2}{2}y'''(t)+C_2y''(t)+C_1y'(t)-\phi[y'(t)]+(1-\alpha)y(t)=0 \tag{A.3}$$

$$\varepsilon\frac{\theta^3}{6}y''''(t)+C_3y'''(t)+C_2y''(t)+C_1y'(t)-\phi[y'(t)]+(1-\alpha)y(t)=0 \tag{A.4}$$

$$\varepsilon\frac{\theta^4}{24}y'''''(t)+C_4y''''(t)+C_3y'''(t)+C_2y''(t)+C_1y'(t)-\phi[y'(t)]+(1-\alpha)y(t)=0 \tag{A.5}$$

Where,

$$C_1=\varepsilon+(1-\alpha)\theta\,;$$

$$C_2=\varepsilon\theta+(1-\alpha)\frac{\theta^2}{2}\,;$$

$$C_3=\varepsilon\frac{\theta^2}{2}+(1-\alpha)\frac{\theta^3}{6}\,;$$

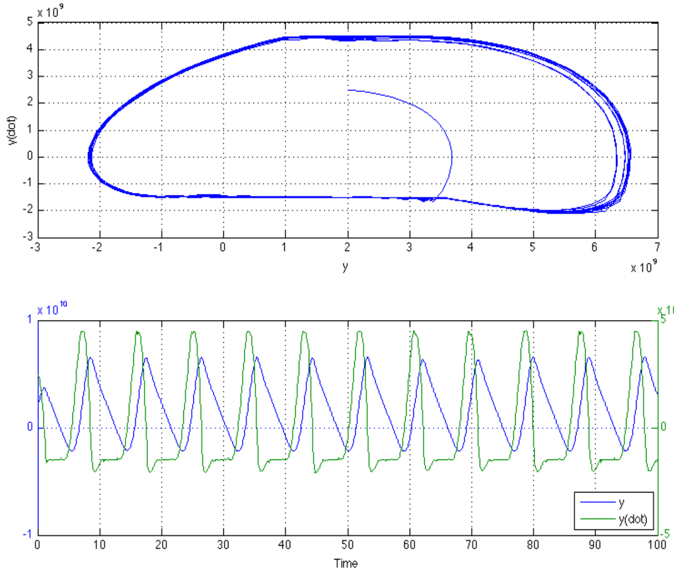
$$C_4=\varepsilon\frac{\theta^3}{6}+(1-\alpha)\frac{\theta^4}{24}\,.$$

The above nonlinear *differential* equations have been simulated, using Matlab, for Goodwin parameters and the results are shown in figure 6. It is very interesting to see how the system behaves when more terms in the Taylor's expansion are retained. Moreover, as the orders increased, the system tended to have more number of solutions and became more *sensitive* to the initial conditions. Commenting on the problem of multiplicity of solutions, Strotz, et.al., insightfully noted that there is not one but at least 25 different limit cycles as solutions for Goodwin's model, and in this exercise, by simulating the model for orders up to 5, we have found that there are at least 45 different limit cycles, all depending upon the initial conditions. The simulational results reinforce the results of Strotz, et.al., and emphasize the need for further understanding of Goodwin's nonlinear model and its fundamental dependence on a 'time-to-build' structure.

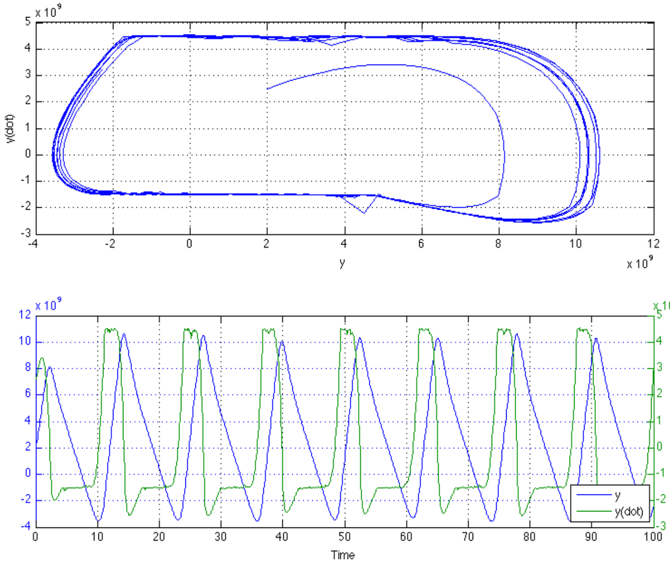
Summary:

> "The problem is to determine what kinds of "initial conditions" lead to the various possible cycles, and then to determine whether these conditions can occur. This presents an analytical problem of great *complexity*, but one that must be solved if non-linear mixed models are to provide unambiguous answers to problems in economic theory."
>
> Strotz, et.al., (1953, p: 408; italics added)

The multiplicity of solutions, depending upon the initial conditions, an economic system can have will decide what kind of possible paths the system can traverse over time. The dependence on the initial conditions and the role of time-to-build function in business cycles theory emphasize the importance of further investigating these nonlinear systems. This exercise is one such attempt to illustrate the richness of the nonlinear difference-differential system that is based on a 'time-to-build' structure used for modelling and analysing macroeconomic dynamics.



1a



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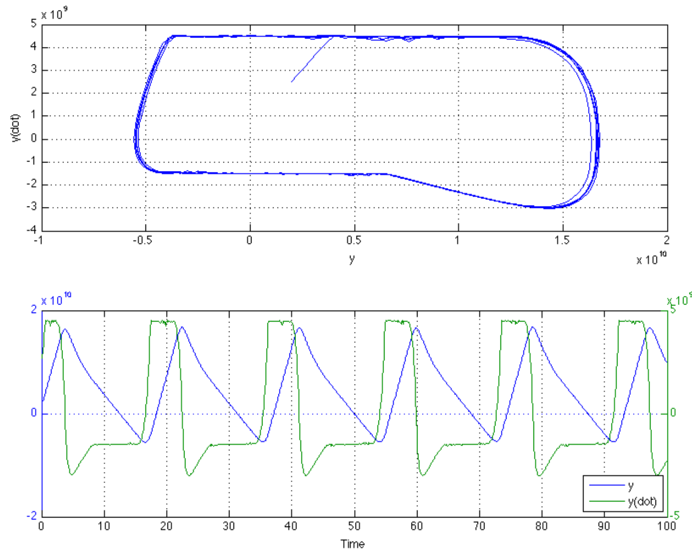

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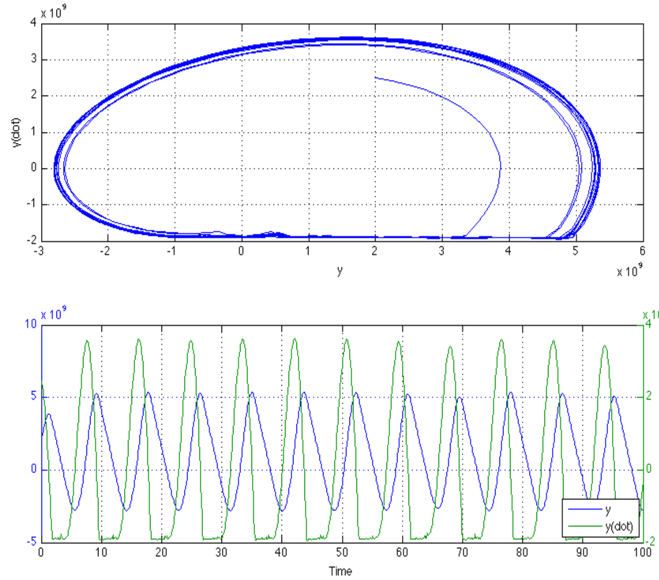

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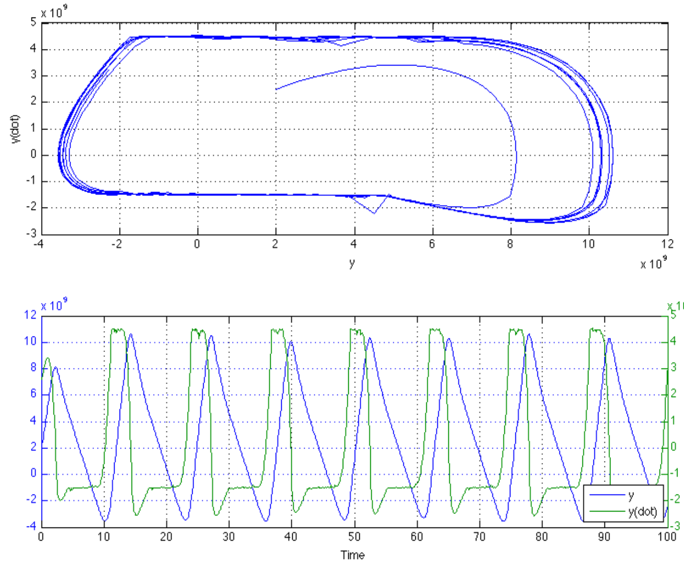

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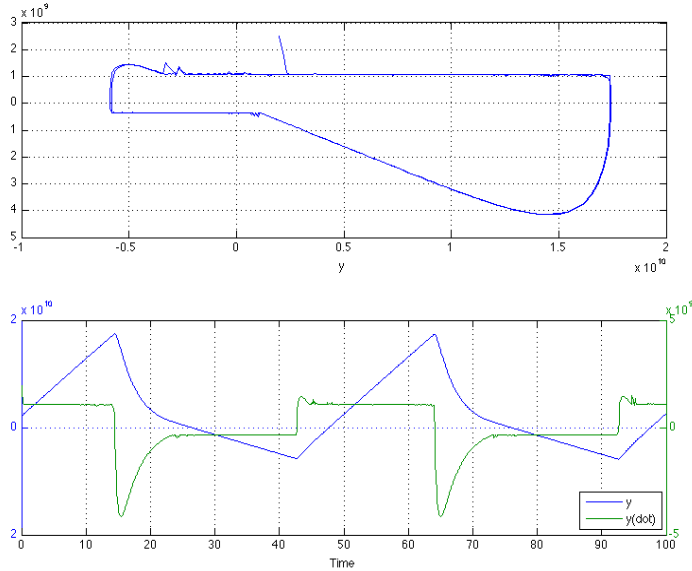

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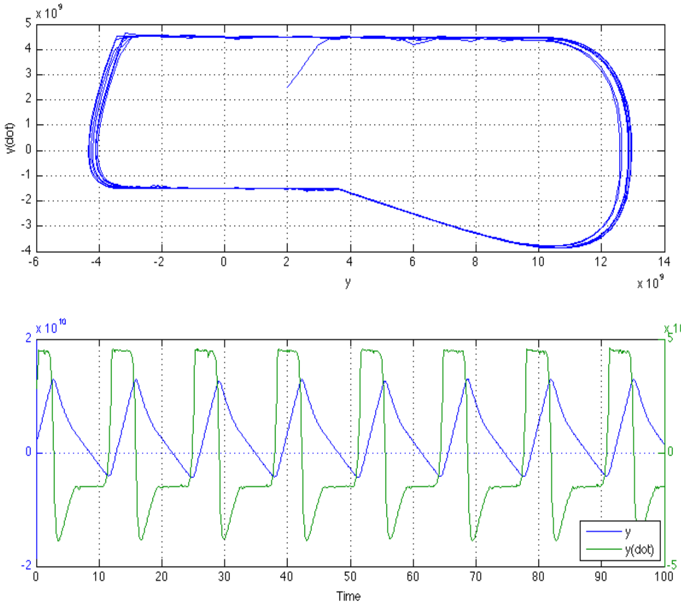
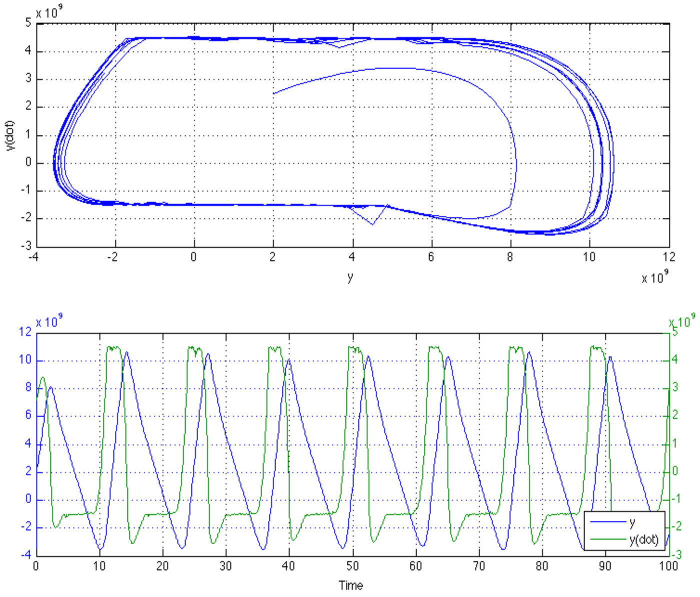


3a



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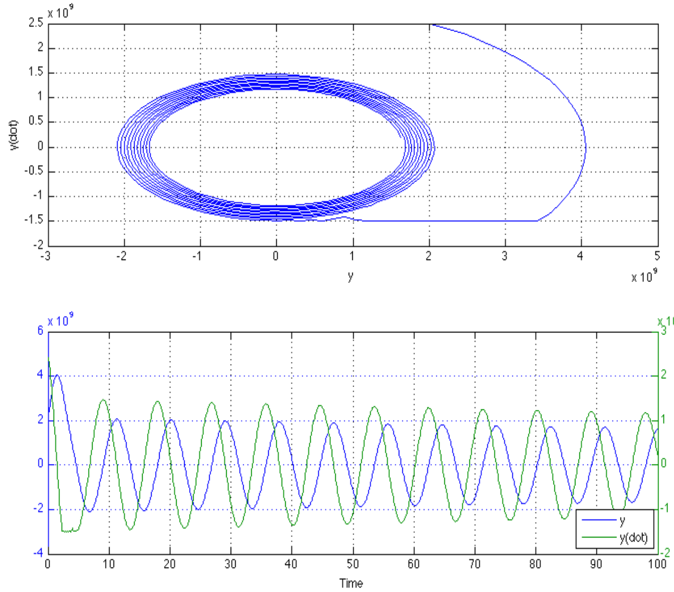

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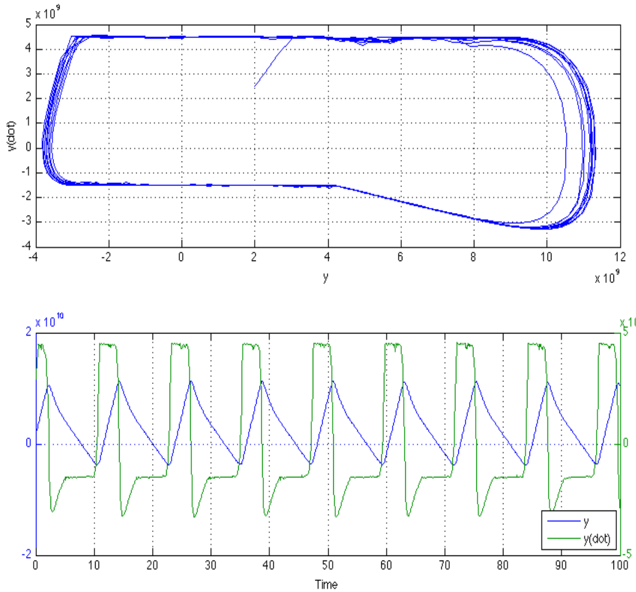

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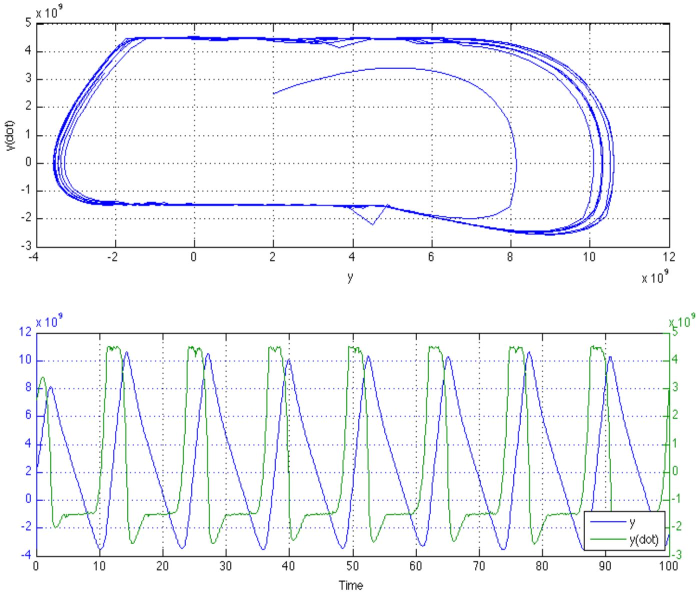

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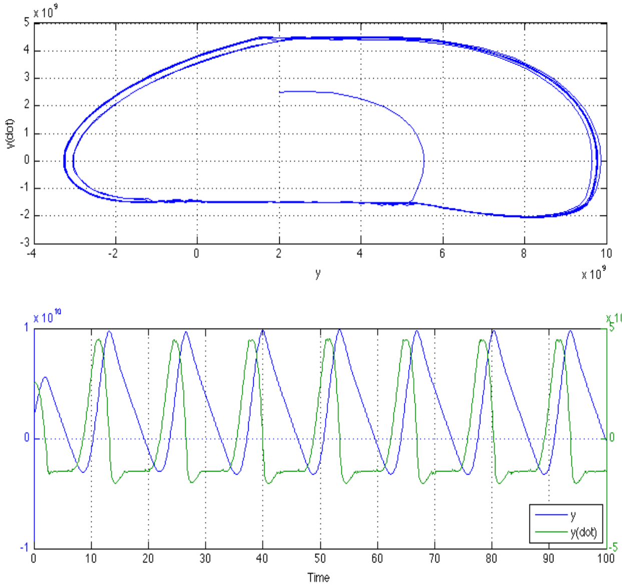

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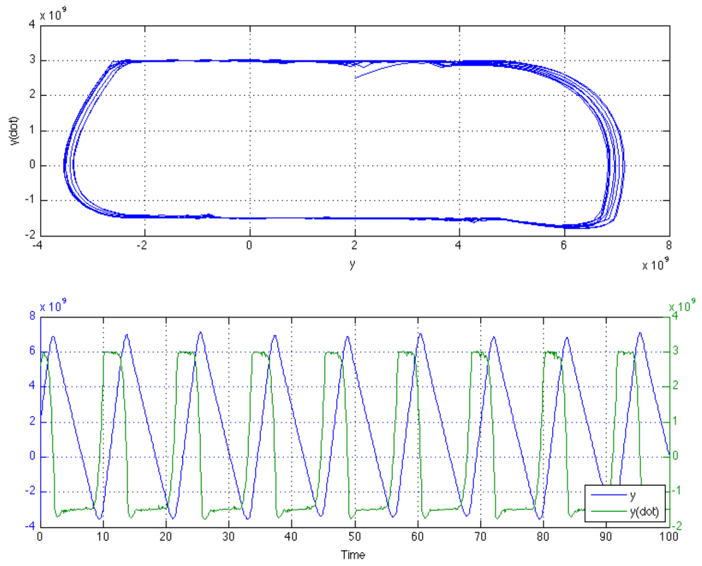
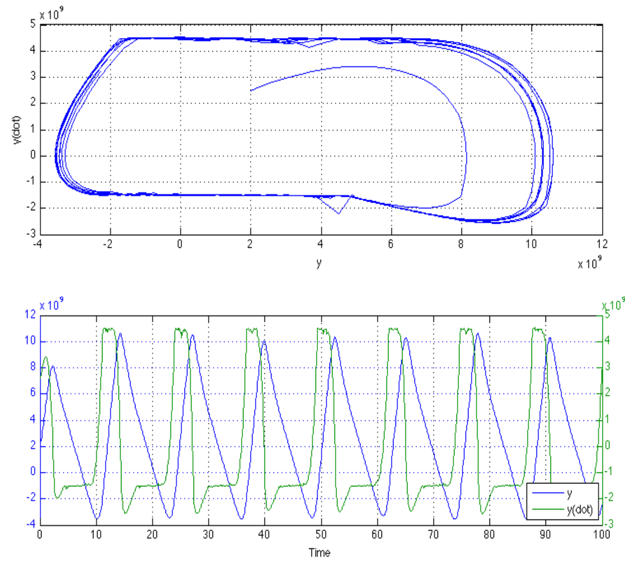
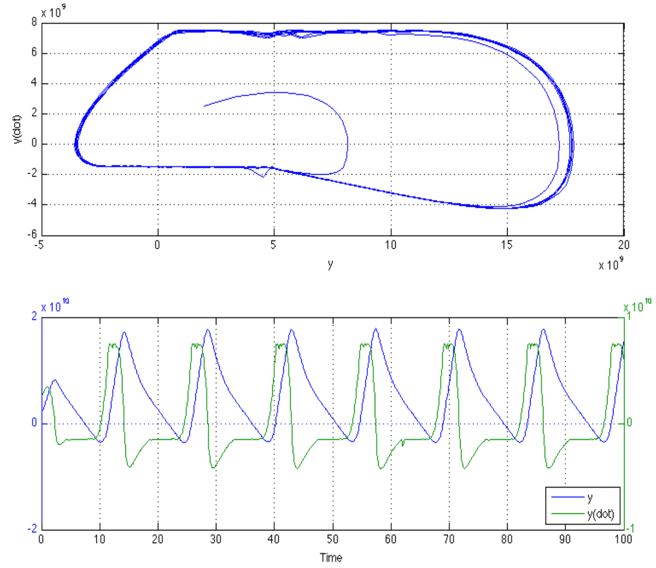


5a



5b



5c

Table 1:

Case	α	κ	ϵ	θ	$\overline{\phi}-\underline{\phi}$
1(b)-5(b)	*0.6*	*2.0*	*0.5*	*1.0*	*9: -3*
1 (a)	0.400	2.00	0.500	1.00	9: -3
1 (c)	0.733	2.00	0.500	1.00	9: -3
2 (a)	0.600	1.58	0.500	1.00	9: -3
2 (c)	0.600	8.42	0.500	1.00	9: -3
3 (a)	0.600	2.00	0.349	1.00	9: -3
3 (c)	0.600	2.00	0.802	1.00	9: -3
4 (a)	0.600	2.00	0.500	0.50	9: -3
4 (c)	0.600	2.00	0.500	1.50	9: -3
5 (a)	0.600	2.00	0.500	1.00	6: -3
5 (c)	0.600	2.00	0.500	1.00	15: -3

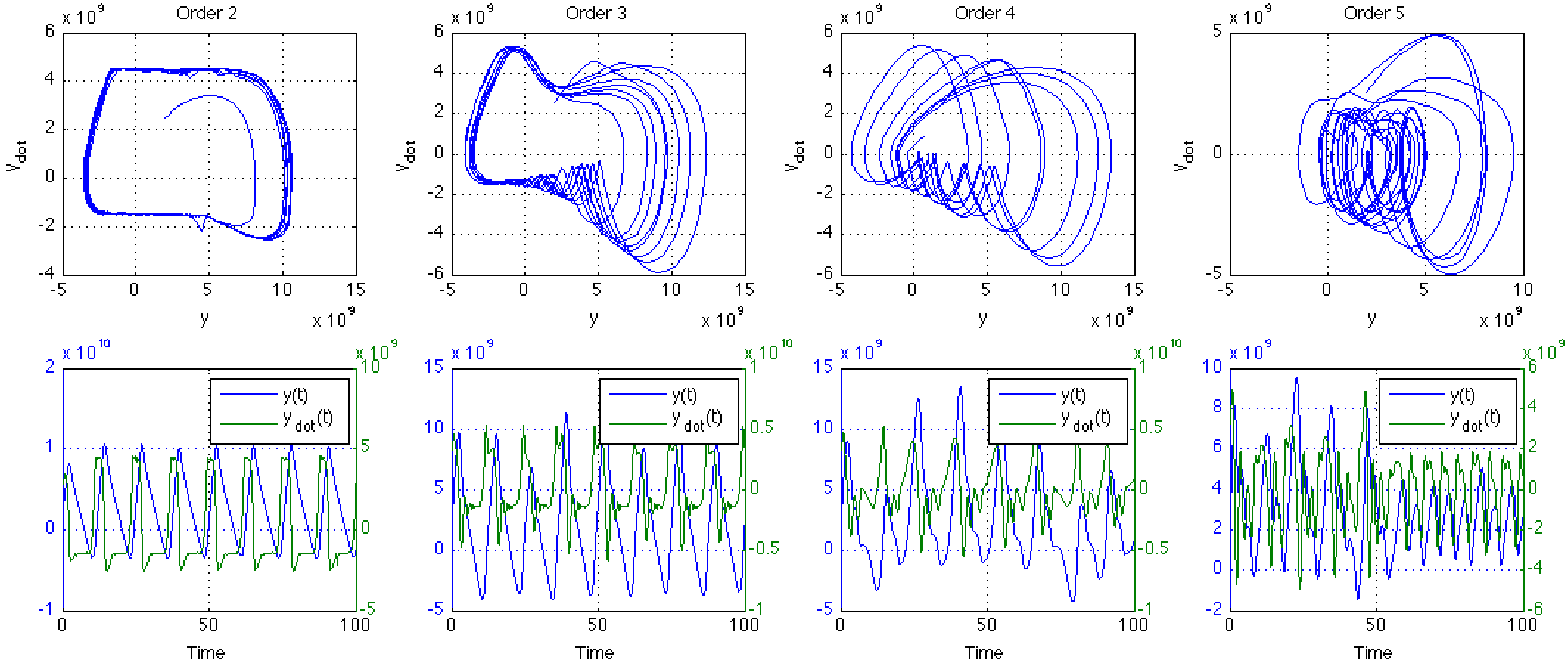


Figure 6

Appendix 2

Leiden, 23-6-1925

Sehr geehrter Herr Professor, –

Gestatten Sie bitte daß ich auf Deutsch schreibe da ich die schwedische Sprache zwar lesen, aber nicht genügend schreiben kann.

Ich bin Student der theoretischen Physik und möchte auf Veranlassung von Herrn Professor Ehrenfest weiter mathematische Nationalökonomie studieren. Bei meinen bisher autodidaktischen Studien in dieser Richtung habe ich aus Ihrem schönen Buch: „Lehrbuch der Nationalökonomie, Theoretischer Teil" sehr viel gelernt. Gerne möchte ich Sie einiges darüber fragen; entschuldigen Sie aber bitte wenn ich durch meine geringe Literaturkentnisse Ihre Zeit unnötig in Anspruch nehme!

Meine Fragen wären folgende:

1. Wie Sie auf S. 101, 102, der deutschen Ausgabe 1913 hervorheben ist der isolierte Tausch ein indeterminiertes Problem, das aber wo Organisationen einander gegenüber stehen doch sehr wichtig ist. Haben Sie oder Andere über dieses Thema vielleicht noch weitere Sachen publiziert die nicht in Edgeworth's „Mathematical Psychics" enthalten sind?

2. Auf S. 152 erwähnen Sie die Gleichung Cournots

über die Preisbildung bei Konkurrenz mehrerer Produzenten. Ist Ihnen außer der Kritik dieser Gleichungen, irgend ein Ausbau derselben oder eine andere Theorie über den Mechanismus der Konkurrenz bekannt, die dergleiche leistet? Ist es nicht von Interesse, diesen Mechanismus weiter zu studieren und denken Sie daß eine weitere Anwendung dieser Gleichungen zu einigermaßen brauchbaren Aussagen über Kartell- und Trusterscheinungen führen kann?

Ich hoffe sehr daß ich Sie nicht störe. Die Wärme mit der Dr. Karl Menger aus Wien, der jetzt in Holland ist, über Sie erzählte, ermutigte mich, diesen Brief an Sie zu richten.

Mit größter Hochachtung
J. Tinbergen

J. TINBERGEN

2566 XD DEN HAAG (L'AJA) 10/2-'84

Dear Professor Velupillai,

Thank you for yours of 3rd February 1984 and for accepting my subject. You will see that I came across an interesting example where another specification than the usual one yields better results.

Thank you also for the three copies of passages in which Einstein and Ehrenfest are mentioned! The anecdote on the queen's visit was wonderful. In fact I know that the queen-mother Emma was much interested in physics. The lectures to the general public given in a hall in The Hague were often attended by her — I saw that as a high-school boy, when I also had access to these lectures.

Unfortunately I lost Wicksell's answer during the war, when a house where we had stored part of my papers was bombed. I remember two things of it. First that I was called by him Herr amanuens (for assistant), but the Swedish dictionary shows that is possible in Swedish. In Dutch an amanuensis only is a laboratory assistant and as a theoretical physicist I could not be an amanuen. Then, Wicksell's suggestion was to read classical authors. This council I did not follow, I must admit. I was more interested in Henry Schultz's type of work, but also in Wicksell's.

Surely, I remember Mademoiselle (in 1938) Godard. She was a very able member of my staff. After the war when she proposed to visit us, we felt that we couldn't receive her and her husband, since all food was rationed and we couldn't offer them a dinner worthy a Swiss couple of their standing. I hope she is still doing well?

With kind regards,

Yours sincerely,

J. Tinbergen