Algorithmic Social Sciences Research Unit

ASSRU
Department of Economics
University of Trento
Via Inama 5
381 22 Trento, Italy

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EXISTENCE PROOFS IN NONLINEAR ENDOGENOUS THEORIES OF THE BUSINESS CYCLE ON THE PLANE – THE ORIGINS[♣]

V. Ragupathy & K. Vela Velupillai

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[♣] This is a sequel to Ragupathy & Velupillai (2012) and in *its* sequel we expect to be able to tackle, more explicitly, the problem of ‘uniqueness’ in nonlinear endogenous theories of the business cycle (NETBC) on the plane. Our indebtedness to the three undisputed pioneers who forged and moulded this field, Richard Goodwin, John Hicks and Nicholas Kaldor, is one that all workers in this field owe to them. The second author, who had the pleasure and privilege of being a formal (and informal) student of Goodwin and Kaldor for his Cambridge doctoral dissertation on the topic of this paper, eons ago, is also more personally indebted to them – and to Hicks, who also became a mentor and friend, but at a slightly later, post-doctoral, stage of his life. Both authors are also deeply grateful to their ASSRU colleagues in Trento, Stefano Zambelli & Kao Selda, and to two of the ASSRU Patrons – Björn Thalberg & Richard Day, also pioneers in NETBC - for years of inspiration, discussion and instruction on the topics discussed in this paper. Needless to say none but the authors are responsible for all remaining infelicities.

1 Introduction

> "The modern business cycle theories in the Keynesian paradigm have a long history. Among them the most relevant ones are of nonlinear types... From the analytical point of view these nonlinear theories, whether verbally or analytically presented, essentially owe the core of their results - that is, the *possibility of business cycles* - either implicitly or explicitly, to the *Poincaré-Bendixson .. theorem* on the *existence* of a limit cycle, except for Goodwin (1967) results, which depend on the Volterra differential equation[1] simulating the symbiosis of prey and predator, that generates infinitely many concentric trajectories"
> Hukukane Nikaido (1996), pp. 217-218 (italics added).

It would not be an exaggeration to state that the tradition of modelling, and proving the existence of, 'cycles' in *Nonlinear Endogenous Theories of the Business Cycle* (henceforth, **NETBC**) *on the plane* (the *phase-plane*), from the late 1940s and the early 1950s, was largely reliant on the use of one or other of the available formal existence theorems, such as, for example, the *Poincaré-Bendixson theorem*[2] (henceforth, *the P-B theorem*). We have claimed (see, Velupillai, 2008) that this tradition originated in the early, pioneering, contribution of Yasui (1952, 1953), Ichimura (1955a,b) and Morishima (1953, 1958). Perhaps the crucial role played by the P-B theorem was first *highlighted* by the important contributions of Rose (1966, 1967,1969), Chang and Smyth (1971).

In a sense the reign of what may be called *the P-B* (and allied existence) *theorems* and the dominance of planar dynamical modelling of NETBC could be said to have lasted – and coincided with – the 'Golden Years of Keynesian Economics', approximately the quarter of a century from 1949 (Goodwin, 1949) to 1973[3].

[1] Of course, the existence of such 'centre-type' attractors for this system, of differential equations also requires a 'proof'. Nikaido, with characteristic candour, goes on (p. 218, italics added):

> "I am myself allied in spirit with the Keynesian paradigm, and will show in this study the possibility of a long-term growth cycle with explicit consideration of both demand-side and supply-side potential based on the [Poincaré-Bendixson] theorem. .... Ignoring the monetary factors here is just for the sake of obtaining a complete growth cycle based on the [Poincaré-Bendixson] theorem."

Nikaido's transparent statement exemplifies the main theme of this paper: an investigation into the way the application of a mathematical theorem determined the nature of the constructed economic model.

[2] To the best of our knowledge, the first time this important theorem was given a pedagogical exposition in an advanced *textbook*, aimed essentially at graduate students in economics, was in (what eventually turned out to be the first of four editions of) Giancarlo Gandolfo's exemplary text on **Mathematical Methods and Models in Economic Dynamics** (Gandolfo, 1971, p.407 & p. 421).

[3] David Gale (1973) is, for us, the fountainhead of the era of NETBC beyond planar dynamics and its underpinnings in *the P-B theorem*, although it took another decade before this was recognised (with the pioneering works of Richard Day (for example, Benhabib & Day,

However both Goodwin (1951) and Hicks (1950) were well aware of the need to prove the *existence* (and *uniqueness*[4]) of endogenously generated aggregate fluctuations in formally acceptable, mathematically rigorous, modes in their economic models of fluctuations. Although one would have expected the distinguished author of **Value & Capital** (Hicks, 1939) to have emphasised this aspect, it was, in fact, Goodwin who was more explicit (ibid, pp. 13-14, italics added):

> "It is intuitively clear that [the aggregate fluctuations] will settle down to [a limit cycle] although *proof* requires the rigorous methods developed by Poincaré. ..... Of another equation [the van der Pol equation] mathematically equivalent to ours [the Rayleigh equation], Andronov and Chaikin say:'Thus while there is no convenient *method for solving* van der Pol's equation, it is known that: (a) there is a unique periodic solution and it is stable; (b) every solution tends asymptotically to *the* periodic solution.' "

Two further points should be noted and emphasised. First of all, there is the important distinction between 'methods of solution' and 'proof of existence of solutions'. Secondly, the classic Andronov & Chaikin (1949) text did, in fact, discuss explicitly *the P-B theorem* for planar dynamical systems (*ibid*, 208-9). The former distinction was a practising *credo* Goodwin maintained in all his work on nonlinear macrodynamics. It is not surprising, therefore, that he did not pay attention to the fact that he could have applied the Poincaré-Bendixson theorem to *prove the existence* of a limit cycle in his (Rayleigh-type) model of nonlinear macrodynamics[5]. The *cognoscenti* would, of course, realise that this concern with existence proofs in NETBC was neither an independent research activity in one isolated field of economics, nor a 'flash in the pan'. Formal concern with existence proofs,using fix point theorems[6], in core areas of economics can be said to have begun with von Neumann (1928) to reach a kind of *zenith* with the Arrow-Debreu classic (Arrow & Debreu, 1954)[7].

1982)). As in all such 'approximate' historical delineations, there is slippage at both ends. Thus, Torre (1977) is an anticipation and Schinasi (1981) and Benassy (1984) are 'hangovers'.

[4]And, indeed, stability, too.

[5]In a personal letter to Velupillai, dated 23 August 1990, Goodwin wrote (italics added):

> "As you are well aware, *I am hopeless at formalism* and I was always pleased to be told by you that what I practice, innocently, is constructive analysis (*proofs being quite beyond me*)."

[6]George Temple (1981, p.119, italics added) made an important observation of the utmost relevance to the general 'vision' underpinning the message in this paper:

> "One of the most fruitful studies in topology has considered the mapping T of a set of points S into S, and the *existence* of fixed points such that $T(x) = x$. The importance of these studies is largely due to their application to ordinary and partial differential equations which can often be transformed into a functional equation $Fx = 0$ with $F = T - I$ where $Ix = x$.

[7]We do not ignore Walras' s valiant – often unjustly belittled – efforts to juxtapose the

In this paper, we examine the role of existence proofs in modelling NETBC as planar dynamical systems. As a by-product we also point out that this can be interpreted as a further attempt to straight-jacket economic theories to existing mathematical results, thereby restricting the attempts at unravelling the real nature of attractor(s) that characterize economic dynamics. Efforts seem to have been directed at adapting economic theories and assumptions to fit those for which existence results concerning the attractors were available.

Thus, the questions we pose are the following: Why did mathematically oriented macrodynamic modellers appeal to existence theorems in the context of business cycles? What were the different existence theorems that were widely used and how? How did these theorems, directly or indirectly, influence the direction in which endogenous business cycle theory proceeded? Are these proofs *constructive*, if not, how can we make them constructive? Are these mathematical objects encapsulating economic phenomena (that is, attractors like limit points and limit cycles) that are *proved to exist*, *computable*?

In order to address these questions, we investigate the role of important *existence* theorems, in particular, the Poincaré-Bendixson theorem (and the Levinson-Smith theorem), in the NETBC models on the plane. The paper is organised in the following way: Section 2 provides a brief overview of endogenous business cycle theory. Section 3 investigates the different existence theorems and the way in which they were invoked using Kaldor's model as an example. Section 4 undertakes a detailed discussion of the use and the impact of Poincaré-Bendixson theorem in NEBCT. In section 5 we discuss the differences between classical existence proofs and that of Poincaré-Bendixson theorem and issue of computability of attractors in this context.

2 Endogenous Business Cycle Theories[8]

The nature and cause of aggregate fluctuations that characterize capitalistic economies has been one of the overriding themes of macroeconomic research and theorizing for more than a century now. Business cycle theory was born as a result of the attempts to incorporate the observed phenomena of cyclical fluctuations into the existing corpus of equilibrium economic theory. Different schools of thought in macroeconomics vary in terms of the sources that they attribute to these fluctuations. Their views can be classified, broadly, into two categories: Those who view that these sources are from 'within' the economic system and those whose view them as being from 'outside'. Consequently, their

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question of existence of equilibrium with devising methods to solve for its determination, in line with the dominant 19^{th} century tradition of algorithmic proofs. In other writings by us (Velupillai, 2009, 2011) these issues have been tackled in greater detail. A comprehensive discussion is forthcoming in Velupillai (2012).

[8]A much more detailed discussion of the topic of this section is in Ragupathy & Velupillai (2012).

theories can be classified as 'endogenous' and 'exogenous' business cycles, respectively. These different approaches have differed in their choice of tools and formalisms while theorizing in a mathematical mode and these choices have in turn influenced the economic assumptions made for these models.

The endogenous view of the business cycles perceives that the aggregate economic fluctuations arise from *within* the system and holds that the fluctuations are a result of an interaction between different economic forces that operate in an economy and it is considered as an intrinsic feature of the system. This is in sharp contrast with the exogenous view, which considers that these fluctuations stem essentially due to factors that are *outside* the system, often termed as exogenous *shocks*, that perturbs the economic system that is, or has a tendency to return to, its equilibrium state. This meant that the formalization of latter was in terms of equations that necessarily had damped roots.

The initial attempts to characterize business cycles in a mathematical mode aimed at building models to demonstrate certain qualitative properties that were observed in the advanced industrial economies at that time. Broadly, these desired properties were: the persistence of economic fluctuations that made the economic system unstable; these upswings and downswings were not symmetrical; the possibility of multiple equilibria was tied together with a widely held belief that the instability is endogenous to the economic system. Characterizing economic fluctuations in a mathematical model with these desired properties meant that there was a need to go beyond linear models, since the latter are capable of exhibiting either damped oscillations, or maintained oscillations only for very specific parameter values. To this end, nonlinearity became an essential feature for these theories, in the mathematical mode. When they started out, the leading theories of business cycles were predominantly endogenous, which later became exogenous. For the endogenous theories, nonlinearity in economic relationships was a crucial element in generating persistent oscillations. Here, we focus on the mathematical apparatus of the endogenous business cycle theories and investigate the role of existence theorems in shaping the mathematization of the economic models in this tradition.

The pioneering works that spearheaded the area of mathematical Nonlinear Endogenous Business Cycle Theory are : Lundberg (1937), Kaldor(1940), Goodwin(1949, 1951) and Hicks(1950). In Kaldor's model, which builds on the works of Harrod (1936), Kalecki (1939) and Keynes (1936), a trade cycle arises due to a dynamic, nonlinear relationship, between investment and savings and, in particular, in the specification of the investment demand function, which responds to the changes in the *level* of capital stock (although a part is also played by an intrinsic nonlinearity in the aggregate savings function). Goodwin and Hicks built on Harrod's work on trade cycles which combines the Keynesian multiplier and the accelerator. Neither was influenced at all by Kalecki, although both Lundberg and Schumpeter (1912, 1939) underpin the foundations of many aspects of Goodwin's approach to NETBC. Hicks developed a piecewise linear multiplier-accelerator model of a growing economy, with the presence of a ceiling and a floor (upper and lower bounds). The economy is constrained by these bounds and continues to oscillate within this corridor. Goodwin(1951) constructed a model using the Keynesian dynamic multiplier and a nonlinear accelerator. Kaldor presented his model in terms of graphical analysis, Hicks modeled his system in discrete time and as a piecewise linear system using difference equations, while Goodwin used difference-differential (which was reduced to a differential equation) equations. However, none of them proved the formal existence theorems concerning these persistent fluctuations. Lundberg's formal models was in the Wicksellian tradition developed by Lindahl, Myrdal and Hammarskjöld and resulted in a piecewise linear discrete model, formally similar to the Hicksian theory. Lundberg, however, resorted to numerical simulations to study the analytical properties of his essentially nonlinear difference equation model (see Velupillai, 2012a for a detailed discussion of the Hammarskjöld-Lundberg tradition, subverted by the unfortunate linearization by Metzler (1941), but revived by the 'traverse' dynamics work of Amendola & Gaffard, 1998).

3 Existence theorems in NEBCT

Though the models of Hicks and Kaldor were nonlinear, they did not succeed in deriving the final nonlinear equation of the model that was necessary to demonstrate the *existence* of sustained fluctuations. It was Goodwin who succeeded in doing so. He reduced his model to a nonlinear second-order differential equation of the Rayleigh-van der Pol type that is capable of exhibiting maintained (relaxation) oscillations. However, he showed the *existence* of the limit cycle for the equation *geometrically* and did not provide an analytical proof of existence. He came very close to hinting at the use of Poincaré's methods, but did not, for reasons given above, make the final 'leap' towards a formal, analytical, proof of existence (see also the previous quote of a part of this passage)[9]:

> "..the system oscillates with increasing violence in the central region, but as it expands into the outer regions, it enters more and more into an area of positive damping with a growing tendency to attenuation. ***It is intuitively clear that it will settle down to such a motion as will just balance the two tendencies, although proof requires the rigorous methods developed by Poincaré.*** It is interesting to note that this is how the problem of the maintenance of oscillation was originally conceived by Lord Rayleigh ... The result is that we get, instead of a stable equilibrium, a stable motion. This concept is the more general one, for a stable equilibrium point may be considered as a stable motion so small that it degenerates into a point. Perfectly general conditions for the

[9]In retrospect, this remark could be considered as envisaging the entry of existence theorems in NETBC.

stability of motion are complicated and difficult to formulate, but what we can say is that any curve of the general shape of $X(\dot{x})$ [or $\varphi(\dot{y})$] will give rise to a single, stable limit cycle."
- Goodwin(1951), pg. 13 [Italics added]

Though Goodwin provided the intuition for the existence and hinted at the direction for proving the maintained oscillations, the final steps were accomplished by the works of the Japanese economists- Yasui, Morishima and Ichimura (roughly in that order; cf., Velupillai, 2008, for a full discussion). Though proving existence theorems was not entirely new to the economic (and game) theorists by then, it was the Japanese trio who were responsible for raising the formal question of *existence* proofs into business cycle theory. Yasui(1953) was interested in formulating the Kaldor model in terms of the van der Pol equation, like the way Goodwin formulated the multiplier accelerator model using the Rayleigh equation. Yasui outlined the methods by which one can cast business cycle models via graphical discussions, especially of nonlinear systems. In this context, he posed the question of formal existence and invoked the (*Levinson-Smith*) *theorem*, thereby bringing questions of existence into the discussions on business cycles.

The subsequent use of *existence* proofs in business cycle theory was by Ichimura, who appealed to the Levinson-Smith theorem once again and discussed the questions of existence in a more detailed manner. The earliest use of Poincaré-Bendixson Theorem in business cycle theory was by Morishima (1953, 1958), where he used both *the Levinson-Smith theorem* and *P-B theorem* in his analysis. Thus, the Japanese trio, made possible the entry of *formal existence* theorems concerning planar differential equations into the business cycle theory, which later went on to crucially determine how nonlinear cycle theory evolved in two dimensions.

For the purposes of this paper, we will focus on the existence theorems related to the following models: Goodwin (1951), Kaldor (1940), Hicks (1950), Kalecki (1935), Lundberg (1937) and their variations. It can be shown that all these models are special cases of the following canonical difference-differential equation.

3.1 Canonical difference-differential Equation

(Bellman & Cooke, 1963, p. 43):

$$\begin{gathered} F[t, u\,(t)\,, u\,(t-\omega_1)\ldots, u\,(t-\omega_m)\,, u'\,(t)\,, u'\,(t-\omega_1)\,, \ldots, \ldots, \\ u^{(n)}\,(t)\,, u^{(n)}\,(t-\omega_1)\,, \ldots, u^n\,(t-\omega_m)] = 0 \end{gathered} \tag{1}$$

This is a n^{th} order difference-differential equation and it is function of $1 + (m+1) + (n+1)$ variables. The functions F and u are real functions, $\omega_i \in \mathbb{R}$ and $n \in \mathbb{Z}$.

3.2 Richard Goodwin:

Goodwin (1951) formulated a nonlinear model of business cycle, in which he combined the nonlinear accelerator with dynamic multiplier, allowing for the presence of investment lags to account for the lag between the time in the enhanced version of the model. The presence of a nonlinearity in the investment function meant that the investment that comes forth is proportional to the change in national income around the unstable equilibrium value. However, capital accumulation becomes highly inflexible once the deviation of the national income from the equilibrium level. He reduces the model to the following equation.

$$\epsilon \dot{y}(t+\theta) + (1-\alpha)y(t+\theta) = O_A(t+\theta) + \phi[\dot{y}(t)] \tag{2}$$

where, y is the aggregate output and ϕ is the nonlinear accelerator, θ is the delay parameter, which accounts for the time-to-build[10] $1-\alpha$ is the propensity to save, O_A is the sum of autonomous consumption and investment outlays and $1/\epsilon$ is the adjustment parameter. Simplifying this we get,

$$\dot{y}(t+\theta) + \frac{1}{\epsilon}(1-\alpha)y(t+\theta) - \frac{1}{\epsilon}O_A(t+\theta) - \frac{1}{\epsilon}\phi[\dot{y}(t)] = 0$$

This is a first order nonlinear difference-differential equation, which is a special case of the equation (1) with 4, i.e, (1+1+2) variables. Goodwin then takes the Taylor series expansion of the terms with time lags $(t+\theta)$ and by retaining the leading terms of the expansion obtains the following second order, nonlinear, differential equation, a special case of the equation (1).

$$\epsilon\theta\ddot{y} + [\epsilon + (1-\alpha)\theta]\dot{y}(t)] - \phi[\dot{y} + (1-\alpha)y = 0 \tag{3}$$

3.3 Mihał Kalecki:

Kalecki (1935) introduces an investment function

$$\frac{I}{K} = \phi\left(\frac{C_1 + A}{K}\right)$$

where C_1, A, K, I are: constant part of the accumulation of the capitalist, gross accumulation equal to the production of capital goods, volume of the existing capital equipment and investment respectively. He then linearizes the function as $I = m(C_1 + A) - nk$ with m, n as the linearization parameters to derive the following final equation:

$$I'(t) = \frac{m}{\theta}[I(t) - I(t-\theta)] - n[I(t-\theta) - U]$$

This can be simplified further as,

$$I'(t) - \frac{m}{\theta}I(t) + \left(\frac{m}{\theta} + n\right)I(t-\theta) + nU = 0 \tag{4}$$

[10]For a detailed discussion of the Time-to-Build Tradition in Business Cycle Modelling, refer Dharmaraj & K. Vela Velupillai(2011)

This is a linear, first order difference-differential equation[11], a special case of the canonical equation (1) as a function of three variables. However instead of linearizing the investment function had he used a nonlinear investment function(see Velupillai,1997), the model would reduced to

$$\frac{K(t)-K(t-1)}{K(t-\theta)} = \phi\left(\frac{C_1+U+\frac{1}{\theta}[K(t)-K(t-\theta)]}{K(t-\theta)}\right) - U$$

This in turn can be simplified and rewritten as

$$K(t)-K(t-1)-\phi(.)K(t-\theta)+UK(t-\theta)=0$$

where $\left(\frac{C_1+U+\frac{1}{\theta}[K(t)-K(t-\theta)]}{K(t-\theta)}\right) = \phi$. This, in turn, is a *nonlinear* difference equation, again a special case of (1), here a function of three variables.

3.4 John Hicks:

Hicks presented his model in terms of linear relationships in discrete time. He starts with the national income identity,

$$Y_n = C_n + I_n + A_n$$

where, $A_n$is the autonomous investment , Y_n, C_n, I_n are total income, consumption and induced investment respectively. By introducing an appropriate number of lags, p, in the presence of the accelerator, the above equation becomes a P^{th} order, linear difference equation.

$$Y_n = A_n + \sum_{r=1}^{p} c_r Y_{n-r} + \sum_{r=1}^{p-1} v_r \left(Y_{n-r} - Y_{n-r-1}\right) + K \tag{6}$$

c_r is the propensity to consume out of the income in period r, or the weight of period p income on the current consumption and v_r is the corresponding accelerator coefficient. This relation above, together with the bounds, that is, a ceiling and a floor that are a result of natural constraints to the system in the form of limited available factors and lower limit of investment, becomes a piecewise linear (hence, nonlinear) P^{th} order difference equation. This, in turn, is a particular case of (1).

[11]In case of Frisch, the model that he presented which was supposed to generate endogenous (non-persistent) fluctuations can be presented as the following second order linear difference-differential equation.

$$\ddot{x} = \left(\frac{s\mu}{\epsilon} - \lambda r\right)\dot{x}(t) + \left(\frac{s\mu}{\epsilon}\right)\dot{x}(t-\epsilon) + \frac{sm}{\epsilon}[x(t)-x(t-\epsilon)] \tag{5}$$

Here, x is the amount of consumer goods produced per year. m, μ, s, ϵ are parameters which stand for: depreciation of capital stock for every unit of consumer good produced, amount of capital stock required for production of one unit of consumer good, the desired cash balance (*encaisse désirée*) parameter for production of capital goods, respectively. However, this model, for the parameter values presented by Frisch do not generate any oscillations as shown by Zambelli(2007).

3.5 Erik Lundberg:

Lundberg's approach was based on the *logic of sequence analysis* and the idea of *cumulative causation*. He constructed *model sequences* in an expanding economy by varying the assumptions on the nature of investment, parameters and initial conditions and studied them for the presence of cyclical behaviour. He worked with a piecewise linear, unstable, model of inventory cycles. This model had built-in natural, economic, constraints acting as bounds that checked the system from unlimited expansion and catastrophic contractions and based on this, the model was made to generate bounded fluctuations. Among the cases he considered, for example, the expansion determined by investment in working capital and fixed capital, respectively, were formulated as second-order difference equations, in terms of the expenditure receipts. Let us consider the case of investment in fixed capital (housing). This was expressed as the following *model sequence*:

$$R_t - R_{t-1} = \left(\frac{\mu(1-\Lambda)}{\sigma}\right)(R_{t-1} - R_{t-2}) + \left(1 - (1-\Lambda)(1 - b - \frac{h-b}{\sigma})\right) R_{t-1} + C \tag{7}$$

R_i - Expenditure receipts from the output of consumer goods at period i, C is a constant autonomous investment in consumer goods inventory, μ is the ratio of income generated from building a house and the sum of expected rent payments during a given period to cover costs, Λ - the propensity to save, $1/\sigma$ is the proportion of total expenditure on consumer expenditure spent on rent payments for housing, h is the proportion of consumer expenditure for dwellings that does not become income during the next period. Simplifying this equation and setting

$$\left(1 + \frac{\mu(1-\Lambda)}{\sigma} + \left(1 - (1-\Lambda)(1 - b - \frac{h-b}{\sigma})\right)\right) = \Theta$$

and

$$\left(\frac{\mu(1-\Lambda)}{\sigma}\right) = \Phi$$

we get the following second order *nonlinear* difference equation, (piecewise linear due to the presence of natural constraints) which is a special case of equation (1) which is a function of 3 variables.

$$R_t - \Theta R_{t-1} + \Phi R_{t-2} - C = 0$$

Lundberg was exploring these *model sequences*, numerically, rather than finding a general solutions for these equations. This exercise can be thought of as a numerical experimentation to identify the possibility of turning points in an expanding economy. This model was later considered by Metzler (1941). He endogenised the bounds and transformed the model into a purely *linear*, second order, difference equation, which again is a very straight forward, special case of (1)

3.6 Nicholas Kaldor

3.6.1 Chang and Smyth:

Let us take the case of Chang and Smyth's exposition of Kaldor's model with nonlinear income and savings functions.

$$\begin{aligned} \frac{dY}{dt} &= \alpha[I(Y,K) - S(Y,K)] \\ \frac{dK}{dt} &= I(Y,K) \end{aligned} \tag{8}$$

Here Y, K, I, S are net income, capital stock, net investment and savings respectively. α is the goods market adjustment parameter. Here the savings and the investment functions are nonlinear. Let us assume, as with Chang and Smyth, $I_Y \equiv \frac{\partial I}{\partial Y} > 0$, $S_Y \equiv \frac{\partial S}{\partial Y} > 0$ and $I_K \equiv \frac{\partial I}{\partial Y} < 0$, $S_K \equiv \frac{\partial S}{\partial Y} < 0$. Differentiating $\frac{dY}{dt}$ and substituting $\frac{dK}{dt} = I(Y,K)$, given the above assumptions, we have

$$\ddot{Y} - \alpha\left(I_Y - S_Y\right)\dot{Y} - \alpha\left(I_K - S_K\right)I(Y,K) = 0$$

This is a second order, nonlinear, differential equation - a special case of (1).

3.6.2 Takuma Yasui:

Yasui casts Kaldor's model into a van der Pol type equation, which is expressed below.

$$\ddot{y} + \frac{1}{\sqrt{\mu\gamma s}}[s + \mu\gamma - \phi'(y)]\dot{y} + y = 0$$

Here, y is deviation of the national income from its stationary or equilibrium value. s is the propensity to save and $1/\gamma$ is the proportion by which the national income changes according to the difference between savings and investment. We will discuss this model in some detail below. The final form of his equation is a continuous time, second order, nonlinear differential equation, a particular case of (1) as well.

Why were the existence theorems used in the first place, in business cycle theory? Is it because the geometrical demonstrations were considered not rigorous enough and therefore an inevitable outcome in the quest for rigour, or was it was purely an accident ? It is possible that the age of existence proofs was dawning upon economics profession, with von Neumann's use of the min-max theorem, the application of Brouwer's fixed point theorem by Nash in game theory and the Arrow-Debreu proof of the existence of the competitive equilibrium, in turn, using Nash's approach. The use existence theorems in business cycle theory by the trio of Japanese economists, who were familiar with nonlinear mathematics and the theory of oscillations, may seem like an inevitable spillover. We will explore the way in which different existence theorems were used in NETBC and the kind of assumptions that were forced to introduce into economic models.

3.7 Kaldor's Model of Trade Cycle

Proving the existence of limit cycles in NETBC for two dimensional autonomous systems has proceeded in two ways and it is useful to distinguish between them. The first approach is to reduce the model of an economy into an autonomous second-order nonlinear differential equation of the Liénard type and appeal to one of the theorems to establish the existence of a unique limit cycle. The second method involves proving the existence of *atleast* one limit cycle for a model which is eventually reduced to a planar dynamical system, by making use of the P-B theorem. Let us begin by considering a slightly modified version of Kaldor model of trade cycles, along the lines of Yasui (1953). We choose this model because it has been studied widely and it was one of the first models in this tradition to which the existence proofs were applied. However, it is equally possible to do the same analysis with the Goodwin's nonlinear accelerator model as well.

We begin with the national income accounting identity,

$$Y = C + I \tag{9}$$

Here, Y stands for aggregate income and C and I stand for consumption and investment, respectively. We now define behavioural equations for the aggregate consumption and investment functions as follows:

$$\begin{aligned} C_t &= c(Y_\tau) + \alpha \qquad (\tau < t) \\ I_t &= \phi(Y_t) - \mu K_t \end{aligned}$$

α for autonomous consumption and the investment function ϕ is assumed to be a nonlinear function. In Kaldor's model, this is an S shaped function. Let us ignore the time subscripts of the variables to keep the notations simple. Taking Taylor series expansion and retaining only the first two terms of the consumption function,

$$C = \beta Y - \gamma \dot{Y} + \alpha$$

Substituting C and I in (1), we get,

$$Y = \beta Y - \gamma \dot{Y} + \alpha + \phi(Y) - \mu K$$

Differentiating the above equation, we get

$$\dot{Y} = \beta \dot{Y} - \gamma \ddot{Y} + \phi'(Y)\dot{Y} - \mu \dot{K} \tag{10}$$

By viewing investment as the change in capital stock:

$$\begin{aligned} \dot{K} = I &= Y - C \\ &= Y - \beta Y + \gamma \dot{Y} - \alpha \\ &= (1-\beta)Y + \gamma \dot{Y} - \alpha \\ &= sY + \gamma \dot{Y} - \alpha \end{aligned}$$

A more illuminating way of looking at this equation, in the context of Kaldor model, is to remember that in the Investment-Savings theory of income determination, the change in income is proportional to the difference between savings and investment.[12] Substituting this in (2), and rearranging the equation, we get

$$\gamma\ddot{Y} + [1 - \beta + \mu\gamma - \phi'(Y)]\dot{Y} + \mu s Y - \mu\alpha = 0$$
$$\gamma\ddot{Y} + [s + \mu\gamma - \phi'(Y)]\dot{Y} + \mu(sY - \alpha) = 0$$

Let us redefine the variables Y and time(t). Write Y in terms of the deviations from the equilibrium(z),

$$z = Y - (\alpha/s)$$

and time as

$$T = \sqrt{\mu s/\gamma}t$$

we obtain a second-order, nonlinear, differential equation.

$$\ddot{z} + \frac{1}{\sqrt{\mu\gamma s}}[s + \mu\gamma - \phi'(z)]\dot{z} + z = 0 \tag{11}$$

Now let,

$$\frac{1}{\sqrt{\mu\gamma s}}[s + \mu\gamma - \phi'(z)] = \zeta'(z)$$

Substituting this in (3), we get a simpler equation,

$$\ddot{z} + \zeta'(z)\dot{z} + z = 0 \tag{12}$$

Here, assume that $\phi'(z) > \mu\gamma + s$ in the neighbourhood of $z = 0$ and this would imply that $\zeta(z)$ is positive or negative, depending on whether the absolute values of z are small or large. Under these assumptions, the equation(3) is the unforced, van der Pol equation and this in turn is a special case of the Liénard equation

$$\ddot{x} + f'(x)\dot{x} + g(x) = 0$$

Note that the van der Pol equation is a special case of this equation and that this equation can be obtained by transforming the Rayleigh equation. Goodwin, for example, reduced his model to an equation of Rayleigh type and demonstrated the existence of limit cycle using the Liénard method[13]. Yasui reduced the Kaldor model to the van der Pol equation, a particular case of the Liénard equation. The idea was to reduce the model under investigation to this canonical, nonlinear equation and invoke the appropriate theorems that ensure the existence of periodic solutions.

It may be useful to clarify the role of nonlinearity in the models of NETBC. Since linear models are also capable of exhibiting oscillatory behaviour, we need

[12] We can express this as follows: $\gamma\dot{Y} = I - sY$

[13] Refer to the appendix for a detailed exposition of the Liénard method.

to distinguish the essential differences in the nature of oscillations between these two and the additional value of placing nonlinearity at the heart of the matter. Linear systems are capable of having periodic solutions (closed paths) if and only if the characteristic equation of these systems have purely imaginary roots. If the roots are purely imaginary, the trace of the coefficient matrix of this characteristic equation vanishes and the system has a center. This means that either all paths are closed, else no path is closed. In contrast, nonlinear models are capable of having *isolated* closed paths, that is, without other closed paths lying next to them. The solution curves that are near wrap themselves around these isolated closed orbits. Closed paths of this type are called as limit cycles. Therefore, nonlinearity becomes a crucial ingredient for constructing endogenous models that are capable of persistent fluctuations.

3.8 The Levinson-Smith theorem for the Kaldor NETBC Model as a Liénard equation

Now we can proceed in three different directions in establishing the existence of limit cycle(s) for the above equation. The first way is to go the old-fashioned geometric way - by transforming the equation by relevant change of variables and study it on the Liénard plane and demonstrate the existence of a limit cycle by constructing it purely geometrically. This is the method resorted by Goodwin and by Yasui. However, Yasui also discussed the applicability of formal existence proofs. The first method is discussed in the appendix. In this section, we analyze the second way, which is to appeal to the Levinson-Smith theorem, like the way the Japanese economists – Yasui, Morishima and Ichimura did, by assuming that certain properties hold for the functions under consideration. The third way is to invoke the P-B theorem, widely used in NETBC, which will be dealt with in the next section.

Yasui succeeded in formulating the Kaldor model in terms of a generalized van der Pol type equation (or Liénard equation) and showed the presence of self-excited oscillations, graphically. It was in this exercise, for the first time, an *existence* theorem was referred to in the theory of business cycles.

> "Thus equation (2.18), in which $\varphi(y)$ and $g(y)$ are assumed to have the above stated properties, is known to be the generalized van der Pol-type equation or Liénard-type equation mentioned above. ... It has been already proved mathematically that in this case (2.18) will have a unique periodic solution"
> -Yasui (1953), pg. 233.[14]

He appealed to the Levinson-Smith theorem which guarantees the existence of a unique limit cycle under certain conditions.

[14] We are using the mimeographed, condensed, version of the original Cowles Foundation Discussion Paper: Economics No. 2065, 1953. This mimeographed version's pagination is from 219-240 and quotations refer to this pagination.

Theorem 1 ***Levinson Smith Theorem***
Consider a two-dimensional differential equation system

$$\dot{x} = y - f(x)$$
$$\dot{y} = -g(x)$$

which is represented as a second-order differential equation,

$$\ddot{x} + f'(x)\dot{x} + g(x) = 0$$

The above equation has a unique periodic solution if the following conditions are satisfied.

1. f' *and* g' *are* C^1
2. $\exists x_1 > 0$ *and* $x_2 > 0$ *such that for* $-x_1 < x < x_2 : f'(x) < 0$ *and* > 0 *otherwise.*
3. $xg(x) > 0 \; \forall x \neq 0$
4. $\lim_{x\to\infty} F(x) = \lim_{x\to\infty} G(x) = \infty$ *where* $F(x) = \int_0^x f'(s)ds$ *and* $G(x) = \int_0^x g(s)ds$
5. $G(-x_1) = G(x_2)$

Proving existence in this case simply becomes an exercise of verifying whether the final nonlinear equation to which the model is reduced does satisfy the conditions required by the above theorem. By assuming functions assumed satisfy the above mentioned properties, the proof of existence of a limit cycle was established. Often, these conditions were too stringent or unrealistic for the economic system to satisfy, so the functional forms were assumed to satisfy these properties. In this case, $\zeta'(z)$ and z are assumed to be C^1 implies that these functions satisfy the *Lipschitz* condition. The last condition is called the symmetry condition and if the functions $f'(x)$ and $g(x)$ to be even and odd functions respectively, this condition is automatically satisfied. This condition plays a crucial role in establishing existence of cycle in this case. Since the van der Pol equation satisfies these conditions, it was possible to prove the existence of a unique limit cycle. This was the practice, for example, in Morishima (1958), Schinasi (1981), Ichimura (1955).

Remark 2 *Note here that the version of the Kaldor's model that we have analyzed assumes that the rate of change of capital stock($\dot{K} = I$) is independent of the level of capital stock and is only a function of income. This assumption is not trivial, since making investment dependent on both capital and income (as it is the case in version investigated by Chang & Smyth (1971)), then the dynamical system does not reduce to Liénard equation (Lorenz(1987), p.286).*

Consequently, it is not possible to apply the Levinson-Smith theorem so long as investment is dependent on capital stock. Alternatively, one can assume that the change in capital stock ($\dot{K}$) is determined by the savings function alone (which is only dependent on income and not on capital stock). For proving existence and uniqueness using theorems other than Levinson-Smith for this model, see Galeotti (1989).

3.9 The Poincaré-Bendixson theorem

Now we analyze the use of another important existence theorem that was very widely used (not just for the theories expressed in terms of Liénard equations) in NETBC.

3.9.1 A Brief Overview of the P-B theorem

The Poincaré-Bendixson theorem is an important *existence* theorem that is used in the study of the qualitative behaviour of the planar dynamical systems and provides positive criterion for presence of limit cycles in the plane.The origin of this theorem dates back to Poincaré, who pioneered the field of the qualitative theory of differential equations. Instead of trying to solve differential equations in terms of explicit solutions, Poincaré developed methods to analyse the qualitative behaviour of solution curves of these differential equations. He thereby developed a geometric approach to understand the global behaviour of these equations on the plane via phase portraits. He classified different kinds of limit sets for these planar differential equations and introduced the concept of 'limit cycles'. We now know that his attempts at an exhaustive classification may not have been successful. He concluded that if the curves do not end up in one of the singular or stationary points, then they are either *closed orbits* or they wrap themselves around these closed orbits. Such limit sets are known as limit cycles.

Later, building on the contributions of Poincaré, the Swedish mathematician Bendixson (1901) proved the same theorem, with much weaker assumptions. The P-B theorem guarantees the existence of limit cycles under certain assumptions, providing the sufficient conditions for its existence on the plane. It provides a precise description of the structure of limit sets in the case of planar dynamical systems and it rules out the possibility of 'chaos' on the plane.[15]

[15]Refer to Ciesielski(2002) for a detailed discussion on its history and its development during the last century.

3.9.2 The P-B theorem for the Kaldor NETBC model as a Liénard equation

Equation (12), which is a second order equation, can be rewritten as a system of two first-order, ordinary differential equations (ODEs) in the following manner:

$$\begin{aligned} \frac{dz}{dt} &= y - \zeta(z) \\ \frac{dy}{dt} &= -z \end{aligned} \tag{13}$$

This system has a unique equilibrium given by (0,$\zeta(0)$). The z and y-nullcline for the above system are

$$y = \zeta(z)\,;\, z = 0$$

respectively. Following Kaldor's assumption that the investment curve is S-shaped, it is clear that $y = \zeta(z)$ has a cubic characteristic and we can divide the $(z - y)$ plane in to four regions.

$$\begin{aligned} V^+ &= \{(z,y) \mid\ y > 0, z = 0\} \\ V^- &= \{(z,y) \mid\ y < 0, z = 0\} \\ g^+ &= \{(z,y) \mid\ z > 0, y = \zeta(z)\} \\ g^- &= \{(z,y) \mid\ z < 0, y = \zeta(z)\} \end{aligned}$$

In the case of the original van der Pol equation, $\zeta(z) = z^3 - z$. It is worth noting that the characteristic of our equation, based on Kaldor's theory also has a cubic characteristic. Therefore, the arguments in our case are analogous to the case of the original van der Pol equation and let us for now assume that $\zeta(z) = z^3 - z$. The Jacobian matrix is the following:

$$\begin{pmatrix} \zeta'(0) & 1 \\ -1 & 0 \end{pmatrix}$$

The eigenvalues, computed from the characteristic equation of this system are:

$$\lambda_\pm = \frac{1}{2}(-\zeta'(z) \pm \sqrt{\zeta'^2 - 4})$$

Given that system has a unique equilibrium $(0, \zeta(0))$, one can analyze the above Jacobian matrix around this point in order to identify the nature of this equilibrium. If $\phi'(z) > \mu\gamma + s$ around $z = 0$, then $\zeta'(z) < 0$ and the equilibrium is a source. Consequently, no solution curve can tend to the equilibrium point over time. It is also the case that any solution curve, which starts in V^+, has to pass through g^-, V^- and g^+ before it enters back[16] to V^+.

Now, define a closed, invariant region on the plane (trapping region) (call it Ω) surrounding the origin, whose boundary is a Jordan curve. Given that the

[16]Refer to Hirsch et.al(2004), Pg. 263-64 for the proof

region $\Omega \subset \mathbb{R}^2$ is closed and invariant with an unstable equilibrium point, one can invoke the following theorem to establish the existence of a periodic orbit. In case of the van der Pol equation, the periodic solution is also a limit cycle- i.e, all the other solutions, except the equilibrium point, tend to this periodic solution.

Theorem 3 ***Poincaré-Bendixson Theorem:*** *Consider a nonlinear autonomous system*

$$\begin{aligned} \frac{dx}{dt} &= F(x,y) \\ \frac{dy}{dt} &= G(x,y) \end{aligned} \tag{14}$$

Let Ω be a bounded region of the phase plane together with its boundary, and assume that Ω does not contain any critical points of the above system. If ϕ is a path of system that lies in Ω for some t_0 and remains in Ω $\forall t > t_0$, then ϕ is either itself a closed path or it spirals toward a closed path as $t \to \infty$. Thus in either case the system has a closed path in Ω.

Here, the vector field all along the boundary of this closed and bounded (hence compact) region points inwards into Ω. This would indicate that the path must spiral towards a closed orbit or it is in itself a closed orbit. The compactness of the space on which these these variables are studied is therefore crucial in ensuring the presence of a closed orbit. In our case, the assumption of compactness of the income space is introduced so as to invoke this theorem. This is even more explicit in the following treatment of the Kaldor's model by Chang and Smyth.

3.9.3 The Kaldor NETBC Model as a Planar Dynamical System and the P-B theorem

There is an alternative way to formalize the Kaldor model, without reducing it in to a Liénard type equation. Instead, it is possible to define relationships between the different variables involved and characterising the direction of changes in one variable with respect to changes in the other. Firstly, *ex-ante* savings and investment are functions of aggregate income and aggregate capital stock. As mentioned earlier, Kaldor's model does not endorse the acceleration principle and it relies on the Savings-Investment theory of income determination. Accordingly, change in aggregate income (Y) is proportional to the difference aggregate savings (S) and investment (I). Investment is defined as the change in capital stock (K) over time. These relations define the following dynamical system.

$$\begin{aligned} \frac{dY}{dt} &= \alpha[I(Y,K) - S(Y,K)] \\ \frac{dK}{dt} &= I(Y,K) \end{aligned} \tag{15}$$

Here we assume nonlinear investment and savings curves as in the original Kaldor model, which twice differentiable, therefore satisfying the *Lipschitz* condition. The partial derivatives, according the assumptions of the Kaldor model can be stated as $I_Y \equiv \frac{\partial I}{\partial Y} > 0$, $S_Y \equiv \frac{\partial S}{\partial Y} > 0$ and $I_K \equiv \frac{\partial I}{\partial Y} < 0$, $S_K \equiv \frac{\partial S}{\partial Y} < 0$. This system can be studied on the Y-K plane and the null clines are given by

$$\begin{aligned} \left.\frac{dK}{dY}\right|_{\dot{Y}=0} &= \frac{S_Y - I_Y}{I_K - S_K} \\ \left.\frac{dY}{dK}\right|_{\dot{K}=0} &= -\frac{I_Y}{I_K} > 0 \end{aligned} \tag{16}$$

The slope of the nullcline $\frac{dK}{dY}\big|_{\dot{Y}=0}$ is greater, less than or equal to zero, depending on whether S_Y is less, greater than or equal to I_Y, respectively. Further, assume that $I_K\, S_Y < S_K\, I_Y$, that is if the slope of the nullcline along which the capital stock is constant, is steeper than the slope of the nullcline along which the income is constant(when it is rising). The Jacobian of this dynamical system is given by

$$J = \begin{bmatrix} \alpha(I_Y - S_Y) & \alpha(I_K - S_K) \\ I_Y & I_K \end{bmatrix}$$

The characteristic roots can be analyzed to understand the nature of the singular point. The assumption that $I_K\, S_Y < S_K\, I_Y$ implies that there is a unique, singular point (y^*, K^*) and assuming that $\alpha(I_Y - S_Y) + I_K > 0$ will ensure that this singular point is unstable node or a focus. By choosing Ω – a compact subset (or assuming that such a compact subset *exists*) in the Y-K plane (non-negative quadrant of $\mathbb{R}^2$), we are ready to invoke the P-B theorem, which guarantees the existence of a closed orbit. Further, if we choose this compact subset Ω in such a way that the closed orbits are in the interior of Ω and assume that the system is structurally stable in Ω, we can establish that these closed orbits are limit cycles.

Proving the existence of limit cycle using this theorem can be synthesized as below.

1. Formulate the dynamic model of the economy as a system of differential equations, encapsulating the nonlinearities in relationships between different economic variables.

2. Reduce the model to a planar (two dimensional) dynamical system.

3. Demonstrate that the economic model thus formulated has a unique equilibrium point.

4. Examine the Jacobian matrix of this system, evaluated at the unique equilibrium point, for the possibility of the equilibrium to be a saddle point and by showing that the determinant of the Jacobian is positive, rule out this possibility. Therefore, the equilibrium can either be a node or a focus.

5. Show that the trace of the Jacobian evaluated at the unique equilibrium point is positive and consequently establish that the unique equilibrium is unstable.

6. Choose an appropriate compact region in the space on which the model is defined in such a way that the chosen compact space is invariant and includes the equilibrium in the interior.

7. Following from the assumption that the chosen set is invariant, establish that the system is structurally stable.

8. Demonstrate that in the boundary of the compact region thus chosen, the velocity vector field points to the interior.

9. Use the Poincaré-Bendixson theorem to prove the existence of at least one limit cycle.

This was also the way it was used by Rose (1967), later by Chang & Smyth (1971), which later became a standard practice in the profession and this was emulated by most other studies mentioned in the previous section. However, it should be noted that not all studies end up demonstrating all the above mentioned points in the strategy and instead simply assume that some of these requirements hold. For example, as Saskura (1994) points out, Schinasi (1982) and Varian (1979) take that point 6 in the above list, i.e, the compact region with the vector field pointing inwards at the boundaries, *exists by assumption*. It is often difficult to demonstrate that such a compact space can be appropriately chosen. Some others assume structural stability of the system in the chosen region, *apriori*.

Remark 4 *The Poincaré-Bendixson theorem guarantees the existence of* ***atleast*** *one limit cycle and therefore uniqueness of this limit cycle is not automatically established. On the other hand, the Levinson-Smith theorem for the Liénard equation does guarantee uniqueness as well. In the case of the van der Pol equation, the uniqueness is established by appealing to the intermediate value theorem. Also, note that the van der Pol equation satisfies the requirements of the Levinson-Smith theorem, which can be invoked to prove existence.*

4 Existence Theorems in NETBC

In this section, we provide a survey of the models of endogenous cycles that use the P-B theorem. Morishima (1958) built on the works of Yasui and Goodwin, attempting to find a way to combine and synthesize the models of Kaldor (Yasui's version) and Goodwin-Hicks, formulated as van der Pol and Rayleigh equations, respectively. The rate of investment was a function of the *level* of income in Kaldor's model and 'rate of change' of income (accelerator) in the model of Hicks-Goodwin. He defined the rate of investment as a linear combination of the investment functions of these two models. He then derived the final equation representing the generalization of these two models and investigated the existence of cycles *formally* by invoking the Levinson- Smith and P-B theorems.

Almost a decade later, Rose (1967) made an important contribution, where he developed a theory of employment cycles in the neoclassical tradition, with profit maximizing firms. In his model, cyclical movements in employment (real effects) arise due to changes in money wages, even in the absence of real balance effects, money illusion and pressure on interest rates. The crucial element that drives the cycles is the nonlinear relationship between employment rate and the wage inflation, that is, the nonlinear phillips curve. He demonstrates the existence of employment cycles by invoking the P-B theorem.

By exploiting this nonlinear relationship capable of generating cycles endogenously, Rose managed to influence NETBC in important ways. First, by taking endogenous cycle theory outside the Keynesian circles to a broader arena, subsequently, forging new ways forward in modeling cycles. Second, his work presented a full fledged invocation and a detailed demonstration of the way in which the Poincaré-Bendixson theorem can be applied in NETBC. By introducing the Poincaré-Bendixson theorem, he provided a strategy for establishing the existence of business cycles in more general conditions than earlier. Further, his works (especially Rose,1969) provided attention to monetary factors in NETBC, which till then was largely focused on 'real' elements of the cycles, except may be for Hicks (1950; see, in particular, Hudson, 1957).

After the contributions by the Japanese economists mentioned earlier, another important contribution to Kaldor's model was made by Chang & Smyth (1971). They undertook a study of Kaldor (1940), addressing questions regarding existence and persistence of cycles. Kaldor's model had nonlinear investment and saving functions, which change according to the direction of capital accumulation, which set the economic system into cyclical motion between stable and unstable equilibrium points. Though the questions of existence were posed in relation to this model by Yasui and Morishima, they did so by reducing the model to a nonlinear differential equations of Liénard type that are known to have stable limit cycles. It was Chang & Smyth, following the path of Rose, who reduced Kaldor's model to a planar dynamical system [17] and showed the existence and persistence of cycles in this system using the P-B theorem.[18]

Kaldor's model was analyzed by Varian (1979) in the light of catastrophe theory that was developed mainly by Thom and Zeeman. In his interpretation

[17]Refer to Semmler(1986) for a survey on nonlinear models, with a focus on limit cycle theories.

[18]It is interesting to note that Rose(1967) discusses the possibility of proving the existence of cycles, albeit in a modified version of the Kaldor's model, adapted for the presence of growth in the system. He outlines the proof strategy in footnote no.1, p.170 for proving the existence of cycles. We don't know whether Chang and Smyth took the hint from here.

of the Kaldor model, he assumes a linear savings function and a nonlinear (sigmoid) investment function, and demonstrates the presence of cyclical behaviour of the system that is locally stable and globally unstable. He introduces the idea of having different rates of change for parameters and the state variables of the functions involved, 'slow' and 'fast' variables respectively, and explains plausible situations where there is a jump in short run equilibrium between different regions of the state space(catastrophes). Depending on whether there are one or two slow variables, the resulting behaviour of the system can be either in terms of 'fold' or 'cusp' catastrophes and the latter is shown to account for the possibilities of slow and fast recoveries in one model. In this framework, Varian uses the P-B theorem to prove the existence of limit cycles.

Research on Hicks' IS-LM model in the context of business cycles also made use of the Poincaré-Bendixson theorem (see Velupillai, 2008a for a more comprehensive discussion of this strand of research). Notable works on this line of research include Schinasi (1981,1982), Benassy(1984) and this came to be called the fix-price macroeconomics approach. Schinasi (1981) in his model of short run fluctuations, combined the dynamic version of the traditional IS-LM model, augmented for the government budget constraint, with the idea of a having a nonlinear investment function. He then showed that this model can be reduced to the Liénard equation and thereby appealing to the Levinson-Smith theorem to prove the existence of a unique limit cycle. In Schinasi (1982), he works in the same framework for the intermediate run, but this time appealing to *the P-B theorem*. The existence proof for the cycles in this model, in particular, the use of P-B theorem was later refined by Sasakura (1994).

Benassy developed a non-Walrasian model, in which business cycles arise due to interaction between 'stabilizing' and 'destabilizing' effects - the former role is played by prices and the latter is due to the unstable accelerator and its resulting quantity dynamics. He works with the IS-LM framework and the traditional Phillips curve (as opposed to the nonlinear Phillips curve like Rose) and incorporates expected demand explicitly into the investment function. The model focuses long run dynamics of the short run (non-Walrasian) equilibrium and proves the *existence* of cyclical behaviour in this case, by invoking the P-B theorem.

In the case of Goodwin's model, Sasakura (1996) and Flaschel (2009) demonstrate the existence of limit cycle under more general conditions than the earlier attempts (using the Liénard method) by using the P-B theorem. The missing case – the application of P-B theorem in proving the existence of cycles in Hicks' trade cycle model, is probably due to the fact that the original Hicks model was formulated in terms of difference equations and as a piecewise linear model, unlike the other two nonlinear models, which were presented in terms of differential equations[19]. However, a recent paper by Matsumoto & Szidarovszky

[19] However, Goodwin's model and Hicks' model are in some sense equivalent, though the

(2010), on a modified version of Hicks' model presents in continuous time with consumption and investment time delays. They make use of the P-B theorem to prove the existence of cycles.

There are many other strands of work in NETBC which make use of the Poincaré-Bendixson theorem – for example numerous studies which focus on inventory cycles, but they do not belong to the 'origins' and, therefore, we do not discuss them here. There will be a place for them in a more extensive, less narrow, study, of which this is only a (beginning) part. It is evident from various studies that were mentioned in this section that the role played by the P-B theorem is both crucial and pervasive across different schools of thought within the nonlinear tradition of endogenous (business) cycles on the plane.

4.1 The Impact of the Poincaré-Bendixson Theorem

As we can observe from the studies mentioned above, this tendency to equate rigour with mathematics of a particular tradition or one kind of mathematics and more specially, proving existence in this case, went beyond the case Walrasian equilibrium to business cycle theory as well. This mode of theorizing meant that, in some sense, methods dictated the questions and hypothesis that were posed and restricted the evolution of this tradition. It also hampered the enlargement of the scope of theories since theorizing concentrated on proving existence, rather than gaining insight into the real nature of the attractors. The questions relating to dynamic methods, processes through which one can understand were ignored in a paradigm that focussed and held 'consistency implied existence' as the mantra.

The introduction of the P-B theorem marked a subtle shift from a modeling strategy in NETBC. Earlier, the strategy was the following: First, reduce the models to some form of Liénard's equation, i.e, nonlinear differential equations like the van der Pol or the Rayleigh equations (which are known to exhibit relaxation oscillations). Then, show that the system has limit cycles, either graphically using the Liénard method as in the case of Goodwin or by appealing to an existence theorem for the uniqueness of limit cycles for the Liénard-type equation. Instead, with the introduction of this theorem, the models were formulated as dynamical systems to investigate the existence of cycles in more general situations, without restrictions on the functional forms as in the earlier case.

One of the important limitations that was posed by the use of the P-B theorem was the constraint on the dimensions of the economic model it demanded. Recall our earlier remark that the vector field has to constantly turning *inwards*

former uses a nonlinear accelerator. This is also why the early work on NETBC by the Japanese economists considered these two models together.

at the boundaries of the compact, invariant space and that there is no equivalent for this theorem in higher dimensions. This has to do with the fact that the P-B theorem in turn depends on the *Jordan curve theorem on the plane*. In the case of a plane, it is easy to decide the direction of the vector field to be either *inside* or *outside* the compact region. Whereas, this is extremely difficult when the dimension is higher than two. For dimensions greater than 2, it can be shown that there is a always a vector that *exists*, that is not in the set of all tangent vectors to the simple closed curve. For example, in the case of a three-dimensional object like a Möbius strip, what is *inside* and *outside* is *undecidable*. This restriction meant that the scope for theorizing was rather limited, since it was not always possible to reduce the dynamic relationships between different economic variables to two dimensions, without making strong assumptions or sacrificing some relevant economic factors which might be crucial in explaining economic fluctuations. To overcome this restriction, NETBC researchers had to either adopt new mathematical tools or to be confined to two dimensions. Eventually, researchers working in the tradition of NETBC reacted by moving on to endorse the tools of bifurcation theory, catastrophe theory and chaos theory.

It should be noted that this theorem is concerned with the qualitative global dynamics of the system. In this framework, it was not possible to assert much regarding the short-run behaviour of the system as economists would desire otherwise. Further, since the theorem proves the existence of *atleast* one limit cycle and the fact that the knowledge of the precise number of limit cycles for the planar dynamical systems is not known in general meant that details regarding the possible kinds (number and nature) of the fluctuations were not available in this mode of theorizing.

The fact that P-B theorem was purely an existence theorem had its impact on NETBC in another important methodological influence for the years that followed. While the use of this theorem may have been decisive in the earlier period of development of NETBC, later models that were developed that were made possible solely because of the availability of this theorem. In other words, economic theory was straight-jacketed to fit the assumptions of these theorems. Once this was the established practice, NETBC moved from depending on one existence theorem to another (those in bifurcation theory, for example) which probably, in retrospect, limited the development of algorithmic methods to study business cycles in NETBC.

The use of the P-B theorem meant that the theory of NETBC was largely restricted to continuous time formalisms, i.e., in terms of differential equations. A corresponding theory of limit cycles, where economic variables exhibit nonlinear relationships between themselves, was not available for difference equations.[20]

[20]There is the other case of difference-differential equations, also known as the delay differential equations, which was used by Frisch, Kalecki (in their *time-to-build* models and by

This also meant that NETBC endorsed continuous time formalisms and topological assumptions needed for the validity of P-B theorem, such as an existence of a compact, invariant space, which might not correspond with either the underlying economic intuition or the nature of the economic quantities which are being studied.[21]

4.2 A Very *Brief* Note on Bifurcation theorems in NETBC

Besides the criterion provided by the above existence theorems for detecting limit cycles, it is also possible to prove the existence of business cycles using bifurcation theorems. We briefly discuss them in this section, without attempting to be comprehensive. Bifurcation refers to instances where parametrized dynamical systems undergo sudden changes in their topological structure. That is, the qualitative behaviour of their trajectories change abruptly at certain points of the parameter space. These points are called bifurcation points and some of the commonly discussed bifurcations include: Poincaré-Andronov-Hopf Bifurcation, Turing bifurcations, pitch fork bifurcation, saddle-node bifurcations, etc. Bifurcation theorems facilitate, in particular, the possibility of working with dimensions higher than two, which is a serious restriction in the case of the *P-B theorem*.

Bifurcation theorems, too, for proving the existence of limit cycles have been widely used in the Keynesian tradition of business cycle theories. Torre (1977) used bifurcation theorems to establish the presence of a stable limit cycle in what he called as the 'Complete Keynesian system', which is structurally stable. However, bifurcation theorems have also been widely used in other traditions of endogenous business cycle theory, for example, to establish competitive endogenous business cycles in overlapping generations models (see, for example, Grandmont,1985).

Within the tradition of endogenous business cycles that this paper has concentrated on, there has been a recent interest in applying bifurcation theorems to Kaldor, Goodwin and Hicks models. Flaschel (2009) summarizes some of the common criteria used for establishing the presence of bifurcations in the appendix to this attractive book. Kaldor's system has been modified to incorporate time-delays and bifurcations occur in these models due to changes in the parameters, capturing either the time delay or the intensity of the effect of the difference between savings and investment on income (see Kaddar and Alaoui, 2008, Krawiec and Szydlowski,1999 and Wang and Wu, 2009). Similar investigations have been carried out for modified versions of Goodwin's model

Goodwin(1951). The latter reduced this to a Rayleigh equation by taking a Taylor series expansion and neglecting higher order terms (which he should not have done, see Dharmaraj & Velupillai, 2011).

[21] Refer to (Velupillai,2010) for arguments against the indiscriminate use of real numbers (and functions defined on real number domains) in economics, since economic quantities can, at best, be rational numbers.

(for example, in Matsumoto, 2009) and Hicks' model (in Matsumoto and Szidarovszky, 2010). By the way, all these modified models are again are special cases of the canonical equation (1).

5 Existence theorems in NEBCT and Computability

5.1 Existence of solutions for IVP of an ODE Vs Poincaré-Bendixson theorem:

While proving existence is fairly straight forward in the case of linear system of differential equations, it is more complicated for nonlinear equations. For some nonlinear equations, there may not be any solutions for certain initial conditions and in some cases there might be many solutions for the same initial condition. For a general case, including nonlinear differential equations, there are additional conditions that need to be imposed and which guarantee (local) existence and uniqueness of solutions. At this point, it is pertinent to wonder why this general existence, uniqueness theorem for solution of ordinary differential equations, a more general theorem, valid for any finite dimension(n), was not used here for establishing the periodic solution. To understand this, we need to look more into the difference between these two theorems.

Theorem 5 ***Existence and Uniqueness of Solutions:*** *Consider the initial value problem*

$$X' = F(x), X(0) = X_0$$

where, $X_0 \in \mathbb{R}^n$. Suppose that F: $\mathbb{R}^n \to \mathbb{R}^n$ is C^1. Then there exists a unique solution of this initial value problem. More precisely, there exists $a > 0$ and a unique solution

$$X : (-a, a) \to \mathbb{R}^n$$

of this differential equation satisfying the initial condition $X(0) = X_0$

In this proof of existence, a sequence of functions are defined via Picard's iteration procedure, by successive approximation of the functions. Later, it is shown that the sequence of functions thus defined would *uniformly converge* to the solution of the given differential equation. Note that F is C^1 implies that the function F satisfies the *Lipschitz condition*, which in turns ensures the contraction to a unique fixed point.

Definition 6 ***Lipschitz Continuity****: Let $D \subset \mathbb{R}^n$ and F: $D \to \mathbb{R}^n$. F is called Lipschitz continuous if for any closed and bounded interval $I \subset D$ there exists a $K \in \mathbb{R}$ and $K < \infty$ with*

$$|F(x) - F(y)| \leq K|x - y|, \forall x, y \in I \tag{17}$$

and K is called the Lipschitz constant.

If the functions $f(x)$ and $g(x)$ in the Liénard equation satisfy the of the *Lipschitz condition*, then, why was the general existence proof for a system of differential equations not used and what is different in the planar case?

Firstly, the above existence theorem, known as the *Picard-Lindelöf theorem*, like the *Cauchy-Peano existence theorem*, is a part of the analytical tradition of studying differential equations. In this approach, the solution of the differential equation is obtained in terms of finding analytic functions equivalent to the differential equation. Though this quantitative approach is valid and accurate, it is quite complicated in practice and often limited in scope. These difficulties led Poincaré to develop the qualitative theory of differential equations. The idea in the qualitative, geometrical, approach is to find methods to understand the qualitative behaviour of the solutions of the differential equations. Theorems such as P-B and Levinson-Smith, are existence theorems in this latter tradition. Therefore, it needs to be clarified that the existence theorems in nonlinear, endogenous theories of business cycles were essentially invoked to establish certain properties of solutions in the spirit of the qualitative tradition.

Secondly, this qualitative approach to understanding dynamics meant that NETBC was an exercise to understand certain global properties of the economic system. The Picard-Lindelöf theorem, however, is a local existence theorem. Initial efforts in NETBC were to establish global behaviour of economic system, in this case – the presence and persistence of cyclical fluctuations, independent of the initial conditions. For planar dynamical systems, the P-B theorem provides a positive criterion for establishing this, in terms of identifying the presence of limit cycles. It is more complicated for higher dimensions and there equivalent for this theorem in higher dimensions (for $n \geq 3$).

Thirdly, the P-B theorem proves the plausibility of *atleast* one limit cycle. However, it does not say anything about the number of limit cycles that are present in the given system and their location, which might be of interest to economists. This leads directly to the unresolved Part B of Hilbert's 16th Problem (see Velupillai, 2008).

5.2 Computability of the attractors and Poincaré-Bendixson Theorem

Besides proving the existence of endogenous business cycles, economists would like study the nature of these cycles, more generally, the properties of the attractors. In the case of planar dynamical systems that characterize the nonlinear models we have discussed, the P-B theorem provides a criterion for the existence of limit cycles. The question that we ask is the following: Is it possible to precisely determine these attractors (in the case of the P-B theorem) and their basins of attraction, in order to gain a deeper insight into the precise nature of the aggregate dynamics of the economy and to undertake meaningful computational experiments? In other words, are attractors of a given dynamical system on the plane *decidable* (*algorithmically*)? Alternatively, can we determine the basin of attraction in which the economy is located at a given time, from the data we have?

Before answering these questions, it would be fruitful to understand why computability of the attractors is important. Studying these attractors to which an economy tends to, and their properties, would have important bearing on theorizing on the kind of policies and control mechanisms to be adopted. As mentioned earlier, *the P-B theorem is an existence theorem, which is proved non-constructively*. This means that the proof does not offer a procedure to identify these attractors, instead, it merely states that such a mathematical object exists. *It might be the case that there exists no finite procedure to determine these attractors.*

Given that the nonlinear dynamical systems are often impossible to solve analytically, simulation experiments to study these systems have relied on numerical methods. Since the above theorem is valid only for a (planar) system of differential equations defined over real number domains, it poses challenges to compute, since not all real numbers are computable. There have been attempts to bridge this gap on two fronts – numerical analysis and computable analysis (and, more recently, also via constructive analysis).

Three questions can be posed at this point.

1. What are the conditions under which we can establish that a solution of the dynamical system and a numerical procedure used to approximate it are equivalent?

2. Can we say something about the possibility of computing the solutions of these dynamical systems from the point of view of computability, instead of numerical analysis?

3. What are the other mathematical tools to gain insight into the nature of the attractors?

To answer the first question, it is worthwhile to note that a numerical procedure is essentially a discrete dynamic object. The numerical procedures – like the Euler method and Runge-Kutta method – can themselves be viewed as dynamical systems. For the solutions approximated by these numerical procedures to be same as that of the original system, it is necessary to show the equivalence between the two and the conditions under which they are so. That is, it is essential to demonstrate that the numerical procedure employed would converge to same invariant sets of the original dynamical system and that structural properties of the original system are retained by the numerical procedure as well. Therefore, the choice of numerical procedures and the conditions under which these procedures faithfully approximate the dynamics of the system which is studied need to considered. Even in this case, it is necessary to discipline these computational investigations using the theory of computability to delineate what can and cannot be computed. These ideas are discussed in Stuart & Humphries (1998) and Velupillai (2010, pp.127-151).

For the second, we need to evaluate the P-B theorem from a constructive and computable point of view and identify the source of the non-constructive or uncomputable content, if there are any. P-B theorem requires that the limit sets are compact subsets of Euclidean space. For example, a classic Compactness criterion for a metric space is given by the Heine-Borel Theorem.

First, a subset X of E is said to be closed if and only if every sequence $\{x_n\}$ of elements of X which converges in E has its limit in X. A subset X of a metric space (E, d) is said to be bounded if there exists a number A such that $d(x, y) \leq A$ for all $x, y \in X$.

Theorem 7 *Heine-Borel Theorem:*
A subset X of $\mathbb{R}^n$ is compact if and only if X is closed and bounded. Alternatively, A subset X of $\mathbb{R}^n$ is compact if and only if each open cover of X has a finite subcover.

This in turn appeals to the Bolzano-Weierstrass theorem, which states that every bounded real sequence in $\mathbb{R}^n$ has a convergent subsequence. This theorem is in turn based on the axiom of completeness for the real numbers. Therefore this kind of definition of compactness is dependent on the completeness axiom of the Cauchy sequences.

Definition 8 *Completeness Axiom*
Every non-empty set of $\mathbb{R}$ which is bounded from above has a least upper bound. Every non-empty set of $\mathbb{R}$ which is bounded from below has a greatest lower bound.

This property does not hold for the set of computable or constructive real numbers.

Theorem 9 *Specker's Theorem:*
A sequence exists with an upper bound, but without a least upper bound.

The Bolzano-Weierstrass theorem is not valid in Constructive Analysis and the *classical* version of the Heine-Borel theorem is not valid in many variants of Computable Analysis (it is however valid in Weihrauch's program with the so-called *type 2 effectivity*). This means that the attractors that are proved to exist by the nonlinear models of business cycles invoking the P-B theorem cannot be computed, unless we have a *computable (or constructive) definition of compact sets*. This is so as long as one works on the domain of real numbers as in the case of the models of NETBC. The question then is whether there is a way to make these models computable?

First of all, we need to restrict the domain of numbers on which we theorize to computable real numbers and work with only computable functions. Results from computable analysis (of the Weihrauch variety) provides some answers. Graça & Zhong (2011) conclude that the attractors and the basins of attractions are *semi-computable* if we assume that the system is stable. In their scheme, they work with so-called type-2 machines. Stability becomes a necessary condition for ensuring the computability of attractors. By employing a notion of computability on closed, open and compact sets as outlined, for example, in Brattka and Weihrauch (1999) and Weihrauch (2000), they are able to prove the following.

Theorem 10 *Let $x' = f(x)$ be a planar dynamical system. Assume that $f \in C^1(\mathbb{R}^2)$ and that the system is structurally stable. Let $K \subseteq \mathbb{R}^2$ be a computable compact set and let K_{cycles} be the union of all hyperbolic periodic orbits of the system, is contained in K. Then, given as input ρ-names of f and K, one can compute a sequence of closed sets $\{K^n_{cycles}\}_{n\in\mathbb{N}}$ with the following properties:*

1. $K^n_{cycles} \subseteq K$ *for every* $n \in \mathbb{N}$
2. $K^{n+1}_{cycles} \subseteq K^n_{cycles}$ *for every* $n \in \mathbb{N}$
3. $\lim_{n\to\infty} K^n_{cycles} = K_{cycles}$

This means that, under the assumption of structural stability, if one can supply the ρ names of f and the compact set K as input, there is an algorithm which can tell, in finite time, whether f has a periodic orbit of the above dynamical system in the compact set K. Since the periodic orbits are only *semi-decidable* in this case, one may need an infinite amount of time, countably calibrated, to conclude that K does not contain a periodic orbit. The same is true for the equilibrium points of the above dynamical system. However, the number of attractors of a given compact set are, in general, undecidable - even if the functional forms are analytic.
All this is part of living with the Halting Problem for Turing Machines and the problem of recursively enumerable sets that are not recursive.

Finally to answer the third question, in the affirmative, we want to mention about the possibilities offered by non-standard analysis (see Velupillai, 2012b).

6 Concluding Notes

The introduction of existence proofs, in particular, the Poincaré-Bendixson theorem, transformed the way mathematical NETBC on the plane envisioned the economics of aggregate fluctuations. In particular, it had an important methodological influence on NETBC, in terms of the mathematical formalisms that the economic theory of aggregate fluctuations embraced and also the role of existence proofs becoming a dominant way of theorizing about economic fluctuations. Whether this development was due to the absence of results in dynamical systems theory or due to the shortcoming of the theorists in terms of developing appropriate mathematical tools for the theoretical problems at hand is not clear. So a categorical judgement on the overall benefit due to the wide acceptance of this theorem, for the development NETBC, is not easy to evaluate - one way or the other. What is undeniable, however, is its importance in the modelling of theories of nonlinear endogenous business cycles.

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8 Appendix -1: Limit cycles on the Liénard plane

> "Therefore, making only assumptions acceptable to most business cycle theorists, along with two simple approximations, we have been able to arrive at a stable, cyclical motion which is self-generating and self-perpetuating. For performing the graphical integration it is convenient, letting $v = \dot{x}$, to rewrite
>
> $$\ddot{x} + X(\dot{x}) + x = 0$$
>
> as
>
> $$vdv + X(v) + x = 0$$
>
> "Thus we have an extremely simple, nonlinear, first order, differential equation, which may easily (**the Liénard method makes it truly easy**) be integrated graphically, provided we have an empirically given $X(v)$ curve. $X(v)$ need not be expressible in any simple mathematical form, although some approximation, say by a cubic expression, does facilitate qualitative discussion of the type of system."
> Goodwin (1951), p. 14, emphasis added.

In this section, we elaborate on the 'Liénard method' and how it makes the graphical integration truly easy, as mentioned by Goodwin in the above quote. The questions that we are interested in exploring are: What is the difference between the normal phase plane and the Liénard plane? What is the Liénard graphical method? What are the advantages of using Liénard plane and how this has been used in the theory of endogenous business cycles?

8.1 Liénard plane:

The early phase of NETBC proceeded in terms of reducing a model of the economy to a canonical nonlinear equation and establishing that the economy was capable of endogenous, self-sustaining oscillations. As we have shown earlier, the two important nonlinear equations in NETBC are the van der pol equation and the Rayleigh equation, which are special cases of a more general equation, namely, the Liénard equation. These equations were then analyzed on a special phase plane (Liénard plane), different from the normal phase plane. By using a special method to construct integral curves on this Liénard plane, it was possible to demonstrate the presence of a limit cycle, graphically.
Let us consider the Liénard equation,

$$\ddot{x} + f(x)\dot{x} + x = 0 \tag{18}$$

Now let us introduce a new variable $v = \dot{x} + F(x)$, where $F(x) = \int_0^x f(s)ds$
We can write the above equation, equivalently, as the following system:

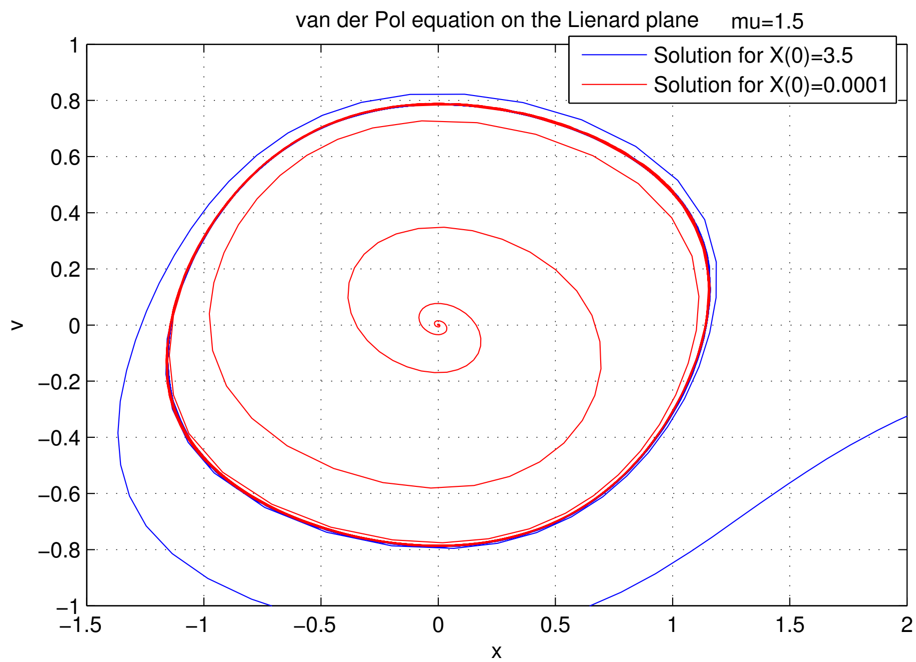
$$\begin{aligned} \frac{dx}{dt} &= v - F(x) \\ \frac{dv}{dt} &= -x \end{aligned} \tag{19}$$

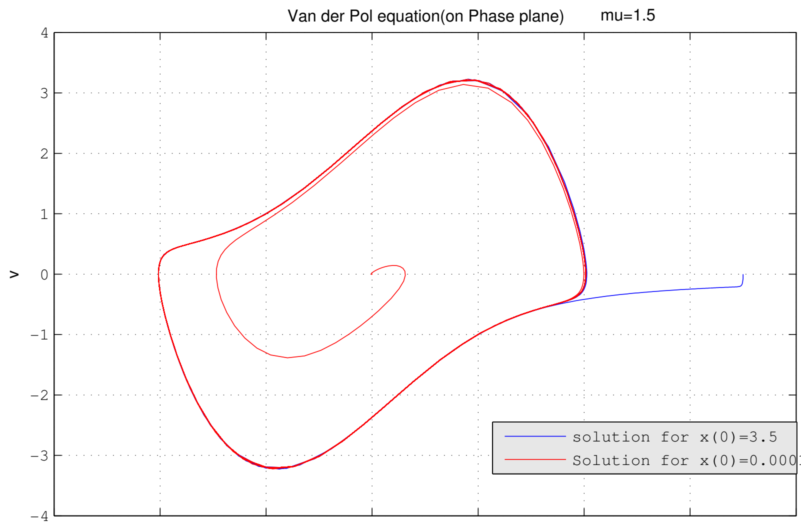
For analyzing the behaviour of this system, let us define a modified plane, Liénard plane:

$$v = \dot{x} + F(x)$$

$$x = -\dot{v}$$

Note that this is different from the ordinary phase plane(x, y), where $y = \dot{x}$. In the ordinary phase plane, the velocity is counted along the vertical axis with respect to the abscissa(x axis). In the Liénard plane(x, v), the velocity is counted along the vertical axis, not with respect to the regular abscissa, but the new 'curvilinear abscissa'. That is, because of the new transformation of co-ordinates that we introduced, $v = \dot{x} + F(x)$, our abscissa has been redefined. Note that the curve $F(x)$ is the characteristic of the equation. Therefore, the trajectories that are traced on the Liénard plane look different from those on the phase plane.





But, what are the advantages of this co-ordinate change, in comparison to the regular phase plane? In order to see this, let us first understand the relationship between the two planes, in terms of the trajectories. There is a one-to-one correspondence implies that the trajectories traced on the Liénard plane can be easily transformed to the ordinary phase plane through the relation,

$$\dot{x} = v - F(x)$$

If the functions $f(x)$ and x are assumed to satisfy the *Lipschitz* condition, existence and uniqueness of solutions are guaranteed on the ordinary phase plane. Since, x is C^1 and $F(x)$ is an integral of a C^1 function $f(x)$, $F(x)$ is also *Lipschitz*. This means that the existence and uniqueness of solutions to the ODE is applicable to the Liénard plane (x, v) as well. This guarantees that the trajectories cannot cross. The one-to-one correspondence, which is continuous both ways, in turn means that the closed path in one plane has a corresponding closed path in the other plane and these qualitative attributes of their solutions do not change across the two planes.
Eliminating time from the above system of equations, we have

$$\frac{dv}{dx} = \frac{-x}{v - F(x)} \tag{20}$$

The above equation gives the equation for the paths of the above dynamical system and in order to analyze these paths, the Liénard plane offers some advantages over the normal phase plane, in the absence of the possibilities of numerical integration using computers. However, this geometric method is not without its advantages and this pre-digital computing era method in fact enabled one of the important discoveries in dynamical systems theory, entirely due to pure macrodynamic motivations: the *one-sided oscillator* with the *Goodwin-characteristic* (for a fairly full discussion of this 'story', see Velupillai, 2008a)

On examining the above equation, we can infer that the paths described by the equation become horizontal when they cross the v-axis. Similarly, only when they cross the curve $v = F(x)$, they become vertical. This feature makes graphical construction of the integral curves easier. Also, if we assume that x and $F(x)$ are odd functions[22], then the equation of the path does not change if the (x, v) are replaced by $(-x, -v)$. This implies that the curve is symmetric with respect to the origin and therefore the paths traced on one half of the plane, then those on the other half of the plane can be obtained by mere reflection by exploiting the property that they are symmetric with respect to the origin. This property of the Liénard plane, in particular, makes it easier, to analyze the paths geometrically.

8.2 Liénard's Method of Graphical Integration:

By setting the variable, $y = \dot{x}$, we can rewrite the Liénard equation as the following:

$$\frac{dy}{dx} + f(x) + \frac{x}{y} = 0 \tag{21}$$

If we change the co-ordinates by introducing $v = y + F(x)$, where $F(x) = \int_0^x f(s)ds$, then the differential equation becomes:

$$\frac{dv}{dx} - \frac{dF(x)}{dx} + f(x) + \frac{x}{v - F(x)} = 0 \tag{22}$$

$$\frac{dv}{dx} + \frac{x}{v - F(x)} = 0 \tag{23}$$

This can be rewritten as

$$xdx + (v - F(x))dv = 0$$

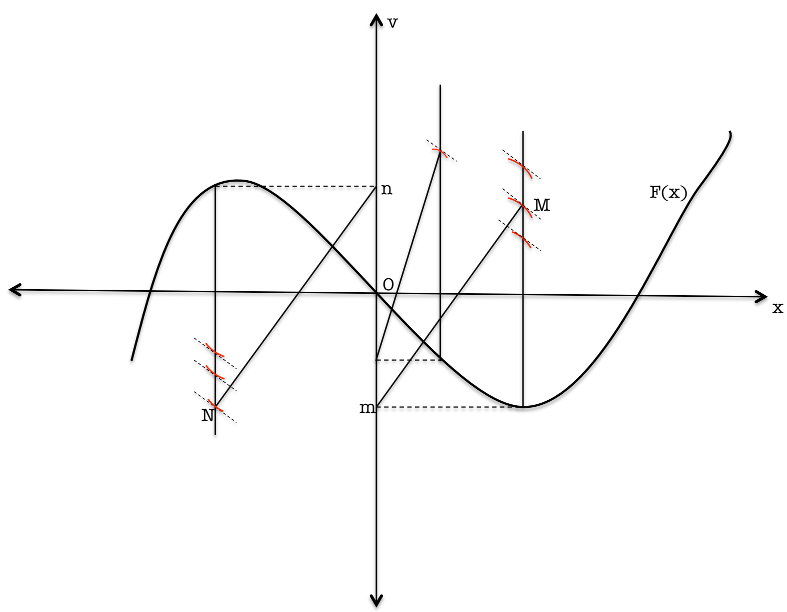
This is nothing but the equation of the normal, with respect to the curvilinear (abscissa) axis and the corresponding ordinate. It is easier to visualize it as the normal passing through $(0, F(x))$:

$$(x - X)dx + (v - V)dv = 0 \tag{24}$$

The procedure to construct integral curves on the plane is the following: Choose any point, say M, on the plane (x,v). In order to construct the curve, we can drop a perpendicular to the curve $F(x)$ and obtain a projection m on the axis parallel to this perpendicular (in the case of the van der Pol equation, the projection would be on the V-axis). Keeping m as the center, one can trace a small arc that passes through M. Similarly, one can trace a family of small arcs keeping m as the center along the perpendicular from M.The line mM gives the normal and it is easy to construct a line perpendicular to mM to obtain the tangent at this point. By repeating the same procedure for different points

[22]Note that if $f(x)$ is assumed to be even, $F(x) = \int_0^x f(s)ds$ becomes an odd function.

and consequently, the projections on V- axis, we can obtain a family of curves. Using this, we can then construct the integral curves and one of these curves in the family will be a closed path, provided the conditions that one imposes on the Liénard equation are met. This, in essence, is the method of graphical integration that was developed by Liénard. On the Liénard plane, given that we know that the paths can be horizontal and vertical when they cross the v axis and the curve $F(x)$ respectively makes this procedure very intuitive and easy. Goodwin uses this method in order to geometrically demonstrate the presence of a limit cycle. Yasui uses the same technique to proceed with graphical integration for a van der Pol type equation. In addition to this geometric demonstration, he formulates his investigation along the lines of existence proofs, stating the conditions that need to be met by the functions in order to formally ensure the existence of periodic solutions. We discuss this in detail in the following section. In contrast, Goodwin's demonstration was purely by geometric construction. In his case, he showed the presence of limit cycle in a Rayleigh type equation, which has a cubic characteristic. Note that the precise shape of the characteristic does not limit this graphical integration method - which is more general and can be applied regardless of the shape of the characteristic.



Construction of Integral curves - Liénard's Graphical Method

8.3 Liénard's criterion for proving existence of limit cycles:

Given the advantages listed in first section above, how can one go about establishing the presence of a unique, stable, limit cycle on the Liénard plane? To see this, we need to introduce the variable that defines the total every of the systemλ as the sum of kinetic and potential energies:

$$\lambda(x,v) = \frac{v^2}{2} + \int_0^x xds$$

$$\lambda(x,v) = \frac{v^2}{2} + \frac{x^2}{2}$$

$$\begin{aligned} \frac{d\lambda(x,v)}{dt} &= \frac{d}{dt}\left[\frac{v^2}{2} + \frac{x^2}{2}\right] \\ &= \frac{d}{dt}\left[\frac{1}{2}(\dot{x} + F(x))^2 + \frac{x^2}{2}\right] \\ &= \dot{x}[\ddot{x} + f(x)\dot{x} + x] + F(x)\frac{d}{dt}(\dot{x} + F(x)) \\ &= F(x)\frac{d}{dt}(\dot{x} + F(x)) \\ d\lambda(x,v) &= F(x)dv \end{aligned} \tag{25}$$

The rationale behind this is given by the Liénard's criterion, which states that for a system to be in a state of sustained oscillation, the change in total energy of the system over a given cycle must be zero. For this the curvilinear integral taken along the trajectory should be zero.

$$\oint F(x)dv = 0$$

First, the condition for the presence of limit cycles can be intuitively inferred from the symmetry condition discussed above. Given that the paths are symmetric to the origin, let us first analyze the length of the two intercepts, along the ordinate ($v - axis$), due these paths traced on the right half of the plane (call them OA and OC). If the length of these intercepts are not equal, then the paths cannot be closed, due to the fact the the paths are symmetric about the origin and the reflection on the other half of the plane would suggest that the paths will never meet if OC is greater or smaller than OA. In order to prove that there is a unique closed path, it would be sufficient to show that intercepts are equal. This is also the idea behind the proof [23] that we invoked a purely geometric criterion for showing the existence of limit cycles in the Kaldor model.

The use of Levinson-Smith theorem in NETBC, by Yasui, Morishima, Ichimura and others, was essentially along these lines - assuming that the relevant economic variables and their functional forms are odd and even, along with other

[23] Refer Minorsky(1962) and Hirsch et.al(2004) for the detailed proof.

requirements of this theorem, guarantees the existence of an unique, stable limit cycle. Symmetry property of the trajectories on the Liénard plane is particularly helpful in the search for closed paths.