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THE TOBIN TAX IN A CONTINUOUS-TIME NON-LINEAR DYNAMIC MODEL OF THE EXCHANGE RATE[♣]

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Abstract

Starting from a new continuous-time non-linear dynamic model of the exchange rate, we formally show that the introduction of a Tobin tax reduces speculators' profit and influences the dynamics of the system making it more stable and less prone to chaotic motion.

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- Keywords: Tobin tax, exchange rate, chaos, continuous-time non-linear dynamics

1 Introduction

It should be clarified at the outset that in this paper the Tobin tax is taken in its original meaning, namely as a tax on all *foreign-exchange transactions,* without considering extensions to all kinds of financial transactions, as is sometimes suggested. Tobin (1974, 1978, 1996)[1] suggested such a tax (with a modest rate) as a means of "throwing sand in the wheels" of international speculation, namely of contrasting speculative capital flows without disturbing medium-long term "normal flows". Such a tax should be applied on all foreign-exchange transactions (both inflows and outflows) independently of the nature of the transaction. This is necessary to avoid the practically insurmountable enforcement problem of distinguishing between foreign exchange transactions for "speculative" purposes and for other purposes. Such a tax, in fact, given its modest rate would not be much of a deterrent to anyone engaged in commodity trade or contemplating the purchase of a foreign security for longer-term investing, but might discourage the spot trader who is now accustomed to buying foreign exchange with the intention of selling it a few hours or minutes later, and who would have to pay the tax every time he buys or sells foreign exchange. A tax of, say, 0.1% (Tobin, 1974, p. 89 originally suggested 1% but later—1996, p. xvii—recommended a lower rate, between 0.25 and 0.1%), namely 0.2% on a round trip to another currency, would cost 48% a year if transacted every business day, 10% if every week, etc., but would be a trivial charge on commodity trade and long-term foreign investment.

In the past the Tobin tax raised much less discussion than it would deserve: in the words of the author himself, "it did not make much of a ripple. In fact, one may say that it sunk like a rock. The community of professional economists simply ignored it" (Tobin, 1996, p. x). Raffer (1998) gives a historical survey of the early debate on the Tobin tax as well as reasons (mainly political, in his opinion) why this debate has been so scanty. A book edited by ul Haq et al. (1996) contains several papers both pro and against. In recent times, however, the Tobin tax has been revived: see, e.g., Hanke et al. (2010), and Xu (2010).

In this paper the Tobin tax is studied by formally introducing it in our non-linear, continuous-time dynamic model for the determination of the exchange rate (Federici and Gandolfo, 2011). The model set forth in the present paper is related to Richard Goodwin's work in two respects:

1) it is non-linear. As is well known, Goodwin was one of the first and most strenuous advocates of the need for non-linear dynamics in economics,

[1] A similar proposal was independently made by Harcourt (1994).

and has built several classic non-linear cyclical and growth-cyclical models as well as non-linear models exhibiting chaos (see the papers collected in Goodwin, 1982, 1989, and Goodwin, 1990). His "conversion" to non-linearity[2] was due to Philippe Le Corbeiller, as he himself tells us: "In one of the early issues of *Econometrica* the French mathematician, Le Corbeiller, produced a brief note in which he told economists that they needed nonlinear dynamics to explain the existence of cycles[3]. Frisch, who was editor of the journal, *must* have read it[4]. What was he thinking when he told Kalecki he had to have exogenous shocks to explain the cycle? The nonlinearities can maintain the cycle, which is what Le Corbeiller was trying to convince economists of; he referred to the fundamental work of van der Pol, which has been of great help to electronic analysis and promised to be so for economists. I cannot imagine why Frisch ignored this advice, and now one can never find out!

One day in the physics lab[5], I was walking down a corridor and I saw a name on the door, "Le Corbeiller". Happily, I remembered that was the name of the man who wrote that article. I had noticed the article but did not understand its significance because, like everyone else, *I too, was thinking in linear terms* [italics added]. I knocked on the door and asked if he was the man who wrote that article: he said, "Yes, come in." He was a kindly, generous man and he helped to educate me in nonlinear dynamics. I would never have arrived without his help". (Goodwin, 1993b, p. 305-306).

2) it is written in continuous time rather than in discrete time. At the beginning of his scientific career in the field of macrodynamics, Goodwin advocated the use of *continuous time* models for several reasons, that it would

[2]In turn, my conversion to non-linearity was due to Goodwin. In the early 1960s, as assistant professor of economics at the University of Rome, I was preparing lecture notes on models of business cycles, and I was baffled by the fact that to explain the persistence of business cycles one had either to assume that exogenous shocks kept alive an otherwise damped cycle (Frisch and others) or to assume that exogenous limits kept into check an otherwise explosive cycle (Hicks and others). Thus such a persistence came to depend on purely exogenous factors, not a very satisfactory explanation. Nowadays everybody knows that this is due to the use of linear dynamical models, which can give rise either to a damped movement or to an explosive movement, the case of a constant-amplitude oscillation being a fluke (and in any case such that the slightest change in the parameters would bring the movement either in the damped or in the explosive region). To borrow Goodwin's statement, I too, was thinking in linear terms. Fortunately I stumbled upon the paper by Goodwin (1951) and by studying it I understood the importance of non-linearity.

[3]The paper that Goodwin is mentioning is Le Corbeiller (1933).

[4]See Ragupathy and Velupillai (2012) for the story of Le Corbeiller's paper.

[5]This refers to the period (1941-45) when Goodwin was teaching physics at Harvard University.

be too long to recall here[6]. He also used discrete-time models[7], but this was made for practical reasons. In this regard, during a personal conversation in Siena I told him that many years before he had convinced me of the superiority of the continuous time approach[8], so that I did not understand his "conversion" to discrete time. He candidly answered that he had not changed his mind, but that he had turned to discrete time for practical reasons, i.e., because it made the numerical simulations of dynamic models much easier, and because to obtain chaos a single (appropriately) non-linear difference equation was sufficient, whereas at least three non-linear differential equations were required. This latter reason was put in writing in his book on chaos (1990, p. vi): "...I have tried to provide examples of a number of different simple models, including both difference and differential equations. Difference equation systems, *though of limited applicability to economics* [italics added], are included because they provide the simplest, reasonably complete introduction to chaotic analysis: in particular, they require no more than one dimension, by contrast with differential equations which require at least three dimensions". However, he never abandoned continuous time, as is shown by the several continuous time models included in that book, and by the fact that his last papers are framed in continuous time (Goodwin, 1991, 1993a).

We shall first give a presentation of the model, and then introduce the Tobin tax.

2 The model[9]

Our starting point is that the exchange rate is determined in the foreign exchange market through the demand for and supply of foreign exchange. This is a truism, but it should be complemented by the observation that, when all the sources of demand and supply—including the monetary authorities through their reaction function—are accounted for, that is, once one has specified behavioural equations for *all* the items included in the balance of payments, the exchange rate comes out of the solution of an implicit dynamic equation.

Let us then come to the formulation of the excess demands (demand minus supply) of the various agents. Our classification is functional. It follows that

[6] Goodwin, 1948, Sect. II. On continuous vs discrete models in economics see Gandolfo, 2009, Chap. 26, Sect. 26.2.

[7] See, for example, Discrete Time and Irregularity, Part III of Goodwin (1989).

[8] I can track this conviction of mine to Goodwin, 1948, Sect. II, a paper that I read in the early 1960s. The conversation took place, if I remember well, in the late 1980s.

[9] This section draws on Federici and Gandolfo (2011, Sects. 2 and 3).

a commercial trader who wants to profit from the leads and lags of trade (namely, is anticipating payments for imports and/or delaying the collection of receipts from exports in the expectation of a depreciation of the domestic currency) is behaving like a speculator.

1) In the foreign exchange market non-speculators (commercial traders, etc.) are permanently present, whose excess demand only depends on the current exchange rate:

$$E_n(t) = g_n[r(t)], \; g_n' \gtrless 0. \tag{1}$$

where $r(t)$ denotes the current spot exchange rate (price quotation system: number of units of domestic currency per unit of foreign currency). Possible transaction costs are subsumed under the non-linear function g_n. On the sign of g_n' see below, Sect. 2.2.

2) Let us now introduce speculators, who demand and supply foreign exchange in the expectation of a change in the exchange rate. According to a standard distinction, we consider two categories of speculators, fundamentalists and chartists[10].

2a) Fundamentalists hold regressive expectations, namely they think that the current exchange rate will move toward its "equilibrium" value. There are several ways to define such a value[11]; we believe that the most appropriate one is the NATREX (acronym of NATural Real EXchange rate), set forth by Stein (1990, 1995, 2001, 2002, 2006). It is based on a specific theoretical dynamic stock-flow model to derive the equilibrium real exchange rate. The equilibrium concept reflects the behaviour of the fundamental variables behind investment and saving decisions in the absence of cyclical factors, speculative capital movements and movements in international reserves. Two aspects of this approach are particularly worth noting. The first is that the hypotheses of perfect knowledge and perfect foresight are rejected: rational agents who efficiently use all the available information will base their intertemporal decisions upon a *sub-optimal feedback control* (SOFC) rule, which does not require the perfect-knowledge perfect-foresight postulated by the Representative Agent Intertemporally Optimizing Model, but only requires *current* measurements of the variables involved. The second is that expenditure is separated between consumption and investment, which are decided by different agents. The consumption and investment functions are derived according to SOFC, through dynamic optimization techniques with feedback control. Thus the NATREX approach is actually an intertemporal optimizing approach, though based on different optimization rules.

[10] For simplicity's sake we neglect the possibility of switch between the two categories.

[11] Typically in the literature the PPP value is used as a measure of the equilibrium exchange rate.

For a treatment of the NATREX, and for an empirical estimation of the $/€ NATREX, see Belloc, Federici and Gandolfo (2008), and Belloc and Federici (2010).

Let us call N_n the nominal NATREX . Then the excess demand by fundamentalists is given by the function

$$E_{sf}(t) = g_{sf}[N_n(t) - r(t)], \quad \mathrm{sgn} g_{sf}[...] = \mathrm{sgn}[...], \quad g'_{sf} > 0. \tag{2}$$

where N_n is the fundamental exchange rate, that we identify with the nominal NATREX, exogenously given and assumed known by fundamentalists. Transaction costs and the like are subsumed under the non-linear function g_{sf}, which is a sign-preserving function

2b) The excess demand by chartists is given by

$$E_{sc}(t) = g_{sc}[ER(t) - r(t)], \quad \mathrm{sgn} g_{sc}[...] = \mathrm{sgn}[...], \quad g'_{sc} > 0, \tag{3}$$

where $ER(t)$ denotes the expected spot exchange rate; the non-linear and sign-preserving function g_{sc} incorporates possible transaction costs. Chartists hold extrapolative expectations:

$$ER(t) = r(t) + h[\dot{r}(t), \ddot{r}(t)], h'_1 > 0, h'_2 > 0, \tag{4}$$

where the overdot denotes differentiation with respect to time, and $h[...]$ is a non-linear function. The assumed signs of the time derivatives mean that agents do not only extrapolate the current change ($h'_1 > 0$) but also take account of the acceleration ($h'_2 > 0$). It follows that

$$E_{sc}(t) = g_{sc}\left\{h[\dot{r}(t), \ddot{r}(t)]\right\}. \tag{5}$$

3) Finally, suppose that the monetary authorities are also operating in the foreign exchange market with the aim of influencing the exchange rate[12], account being taken of the NATREX, by using an integral policy *à la* Phillips. The authorities' excess demand $E_G(t)$ can be represented by the following function:

$$E_G(t) = G\left\{\int_0^t [N_n(t) - r(t)]\, dt\right\}, G' \gtrless 0. \tag{6}$$

where $G\{...\}$ is a non-linear function and the integral represents the sum of all the differences that have occurred, from time zero to the current moment, between the NATREX and the actual values of the exchange rate. The sign of

[12] Central banks have often used direct interventions as a tool to stabilize short-run trends or to correct long term misalignments of the exchange rate.

G' depends on the policy stance of the monetary authorities. More precisely, if the aim is to stabilize the exchange rate around its NATREX value, then $G' > 0$. In fact, in such a case,

$$sgnG = sgn \left\{ \int_0^t [N_n(t) - r(t)] \, dt \right\}, \tag{7}$$

because if the sum of the deviations is positive, this means that the NATREX has been on average greater than the actual exchange rate, so that the latter must increase (depreciate) to move towards the NATREX, hence a positive excess demand for foreign exchange. The opposite holds in the case of a negative sum. Thus the function G passes through zero when moving from negative to positive values, and $G'(0) > 0$.

But the authorities might wish to maintain or generate a situation of competitiveness, which occurs when the actual exchange rate has been on average greater than the NATREX, hence the integral is negative. To maintain or accentuate this situation, the authorities demand foreign exchange, so that

$$sgnG = -sgn \left\{ \int_0^t [N_n(t) - r(t)] \, dt \right\}. \tag{8}$$

Thus the function G passes through zero when moving from positive to negative values, and $G'(0) < 0$.

Market equilibrium requires

$$E_n(t) + E_{sf}(t) + E_{sc}(t) + E_G(t) = 0. \tag{9}$$

In this way we have only one endogenous variable, $r(t)$, since the fundamentals are subsumed under the NATREX, which is known to both the authorities and the fundamentalists, and is considered exogenous in the present model.

Since the market equilibrium condition (9) holds instantaneously (given the practically infinite speed of adjustment of the FOREX market), we can differentiate Eq. (9) with respect to time, thus obtaining

$$\dot{E}_n(t) + \dot{E}_{sf}(t) + \dot{E}_{sc}(t) + \dot{E}_G(t) = 0. \tag{10}$$

By differentiating Eqs. (1), (2), (5), and (6)[13] with respect to time and substituting the result into Eq. (10) we obtain

$$g'_n \times \dot{r}(t) + g'_{sf} \times [\dot{N}_n(t) - \dot{r}(t)] + g'_{sc} \times [h'_1 \times \ddot{r}(t) + h'_2 \times \dddot{r}(t)] + G' \times [N_n(t) - r(t)] = 0. \tag{11}$$

[13] Note that $\dot{E}_G(t) = G' \times [N_n(t) - r(t)]$.

Collecting terms we get

$$g'_{sc}h'_2\times\dddot{r}(t)+g'_{sc}h'_1\times\ddot{r}+(g'_n-g'_{sf})\times\dot{r}(t)-G'\times r(t)=-G'\times N_n(t)+g'_{sf}\times\dot{N}_n(t), \tag{12}$$

whence, dividing through by $g'_{sc}h'_2\neq 0$,

$$\dddot{r}(t)+\frac{h'_1}{h'_2}\ddot{r}+\frac{(g'_n-g'_{sf})}{g'_{sc}h'_2}\dot{r}(t)-\frac{G'}{g'_{sc}h'_2}r(t)=-\frac{G'}{g'_{sc}h'_2}N_n(t)+\frac{g'_{sf}}{g'_{sc}h'_2}\dot{N}_n(t). \tag{13}$$

This equation may seem linear, but it is not so. In fact, the derivative of a function is a function of the same arguments of the function, i.e.

$$g'_n=f_n[r(t)],g'_{sf}=f_{sf}[N_n(t)-r(t)],g'_{sc}=f_{sc}\left\{h[\dot{r}(t),\ddot{r}(t)]\right\},\quad \text{etcetera}, \tag{14}$$

so that the coefficients of Eq. (13) are to be considered as (non-linear) functions.

The model could be linearised and the resulting linear form analysed, but this would be uninteresting in the present context, since a linear model cannot give rise to chaos. The problem then arises of specifying the non-linearities of our model.

2.1 The intrinsic non-linearity of the model

When one abandons linearity (and related functional forms that can be reduced to linearity by a simple transformation of variables, such as log-linear equations), in general it is not clear which non-linear form one should adopt. Further to clarify the matter, let us distinguish between *purely qualitative* non-linearity and *specific* non-linearity.

By *purely qualitative* non-linearity we mean the situation in which we only know that a generic non-linear functional relation exists with certain *qualitative* properties, such as continuous first-order partial derivatives with a given sign and perhaps certain bounds. This is the aspect so far taken by our model, but it is hardly useful for our purposes, because the econometric estimation obviously requires specific functional forms.

By *specific* non-linearity we mean the situation in which we assume a specific non-linear functional relationship. Since in general it is not clear from the theoretical point of view which non-linear form one should adopt, the choice of a form is often arbitrary or made for convenience.

In our case, however, it is possible to introduce a non-linearity on sound economic grounds. This concerns the excess demand of non-speculators. To understand this point, a digression is called for on the derivation of the demand and supply schedules of these agents.

2.2 Derivation of the demand and supply schedules of non-speculators [14]

The main peculiarity of these demand and supply schedules for foreign exchange is the fact that they are *derived* or *indirect* schedules in the sense that they come from the underlying demand schedules for goods (demand for domestic goods by nonresidents and demand for foreign goods by residents). In other words, in the context we are considering, transactors do not directly demand and supply foreign exchange as such, but demand and supply it as a consequence of the underlying demands for goods. Thus the demand for and supply of foreign exchange depend on the elasticities of the underlying demands for goods. Consider for example $S(r)$, the total revenue of foreign exchange from exports (determined by export demand), which depends on the elasticity of export demand. If the elasticity of exports is greater than one, an exchange-rate depreciation of, say, one per cent, causes an increase in the volume of exports greater than one per cent, which thus more than offsets the decrease in the foreign currency price of exports: total receipts of foreign exchange therefore increase. The opposite is true when the elasticity is lower than one.

Since a varying elasticity is the norm rather than an exception (a simple linear demand function has a varying elasticity), cases like those depicted in Fig. 1 are quite normal.

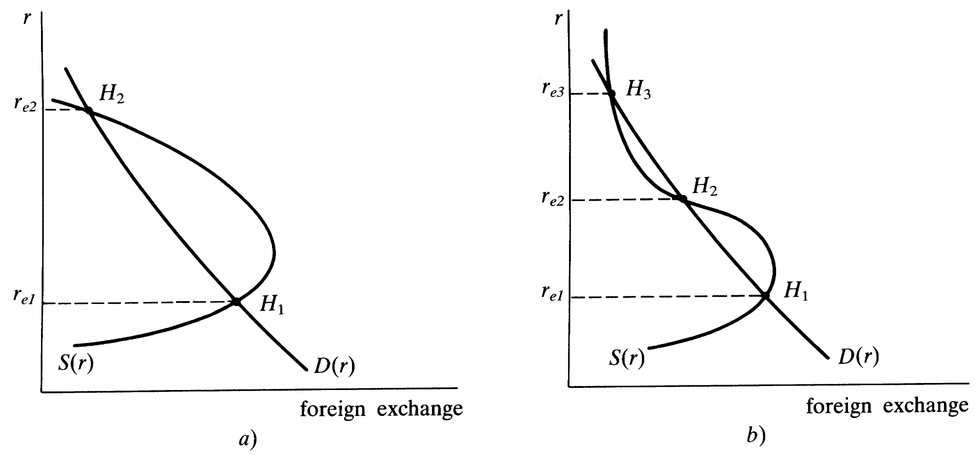

Figure 1: Non-linear supply curves

In the case depicted in Fig. 1a) the function $S(r)$ can be represented by

[14] For an in-depth treatment of this point see Sect. 7.3.1 of Gandolfo, 2002.

a quadratic, while in the case of Fig. 1b) a cubic might do. Let us consider the simpler quadratic case, $S(r) = a + br + cr^2, a > 0, b > 0, c < 0$,where a, b, c are constants[15].

2.3 The final non-linear jerk equation

What we propose to do is to introduce the above quadratic non-linearity *while assuming all the other functions to be linear and with constant coefficients.* Thus, assuming that $D(r)$ is linear ($D(r) = d_0 + d_1 r, d_0 > 0, d_1 < 0$, where d_0, d_1 are constants), we can write

$$E_n(t) = D(r) - S(r) = (d_0 + d_1 r) - (a + br + cr^2) = (d_0 - a) + (d_1 - b)r - cr^2 \quad (15)$$

Given this, we have

$$\dot{E}_n(t) = \alpha \dot{r}(t) + \beta r(t)\dot{r}(t), \qquad \text{where } \alpha = (d_1 - b) < 0, \beta = -2c > 0. \quad (16)$$

Comparing Eqs. (15) and (1) we note that

$$g'_n = (d_1 - b) - 2cr(t) = \alpha + \beta r(t). \quad (17)$$

As regards the other excess demands, we set

$$\begin{aligned}
E_{sf}(t) &= m[N_n(t) - r(t)], m = g'_{sf} > 0 \\
E_{sc}(t) &= n[ER(t) - r(t)], n = g'_{sc} > 0 \\
ER(t) &= h[r(t), \dot{r}(t), \ddot{r}(t)] = r(t) + b_1 \dot{r}(t) + b_2 \ddot{r}(t), \quad (18) \\
h'_1 &= b_1 > 0, h'_2 = b_2 > 0; \text{ replacing in the previous equation we get} \\
E_{sc}(t) &= nb_1 \dot{r}(t) + nb_2 \ddot{r}(t) \\
E_G(t) &= g \left\{ \int_0^t [N_n(t) - r(t)]\, dt \right\}, g = G' \gtreqless 0
\end{aligned}$$

where m, n, b_1, b_2, g are all constants. Substituting Eq. (17), and the parameters defined in (18), into Eq. (13), and rearranging terms, we obtain

$$\dddot{r}(t) + \frac{b_1}{b_2}\ddot{r}(t) + [\frac{\alpha - m}{nb_2} + \frac{\beta}{nb_2} r]\dot{r}(t) - \frac{g}{nb_2} r(t) = \frac{-g}{nb_2} N_n(t) + \frac{m}{nb_2}\dot{N}_n(t) = 0, \quad (19)$$

[15]We have chosen the quadratic form for simplicity's sake and on the basis of the parsimony principle. Besides, running a quadratic and a cubic interpolation on the data for $S(r)$ and r did not give substantially different results.

The quadratic function $a + br + cr^2$ as represented in the diagram implies $a > 0, b > 0, c < 0$.

or

$$\dddot{r}(t) = -\frac{b_1}{b_2}\ddot{r}(t) + [\frac{m-\alpha}{nb_2} - \frac{\beta}{nb_2}r(t)]\dot{r}(t) + \frac{g}{nb_2}r(t) + \varphi(t), \tag{20}$$

where

$$\varphi(t) \equiv -\frac{g}{nb_2}N_n(t) + \frac{m}{nb_2}\dot{N}_n(t). \tag{21}$$

The homogeneous part of the non-linear third-order differential equation (20) is a jerk function[16], and is known to possibly give rise to chaos[17] for certain values of the parameters [Sprott, 1997, eq. (8)]. This possibility has been explored in our previous paper (Federici and Gandolfo, 2011) where the model has been estimated and simulated.

For estimation purposes the final jerk equation can be written as

$$\dddot{r}(t) = a_1\ddot{r}(t) + [a_2 + a_3 r(t)]\dot{r}(t) + a_4 r(t) - a_4 N_n(t) - a_5\dot{N}_n(t). \tag{22}$$

where

$$\begin{array}{l} a_1 \equiv -\frac{b_1}{b_2} < 0, \\ a_2 \equiv \frac{m-\alpha}{nb_2} > 0, \\ a_3 \equiv -\frac{\beta}{nb_2} < 0, \\ a_4 \equiv \frac{g}{nb_2} \gtrless 0, \\ a_5 \equiv -\frac{m}{nb_2} < 0. \end{array} \tag{23}$$

The expected signs of the a_i coefficients reflect our theoretical hypotheses set out in the previous sections. We note that the "original" parameters are seven $(b_1, b_2, m, \alpha, n, \beta, g)$ while we can estimate only five coefficients. Hence it is impossible to obtain the values of the original parameters. What we can do is to check the agreement between the signs listed in (23) and the coefficient estimates. The estimates[18] are reported in Table 1, and have been extensively examined in Federici and Gandolfo (2011).

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[16] A jerk function has the general form

$$x''' = F(x'', x', x).$$

In physical terms, the jerk is the time derivative of the acceleration.

It seems that the denomination "jerk" came to the mind of a physics student traveling in a car of the New York subway some twenty years ago. When standing in a subway car it is easy to balance a slowly changing acceleration. But the subway drivers had a habit of accelerating erratically (possibly induced by the rudimentary controls then in use). The effect of this was to generate an extremely high jerk.

[17] As pointed out by Professor Velupillai, there is, at the moment, no theory of chaos - although there are definitions of chaos. Here we follow Sprott (1997) in defining chaos as a sequence of values that exhibit sensitive dependence on initial conditions and long term

Table 1: Estimation results

Coefficient	Estimate	ASE	Ratio
a_1	-12.405	1.538	8.06
a_2	16.976	2.823	6.01
a_3	-27.421	3.545	7.73
a_4	-0.01064	0.003596	2.96
a_5	-1.226	0.184	6.66
Log-likelihood value	0.3287539E+05		

3 A Tobin Tax

There are several ways (not mutually exclusive) of modelling a Tobin tax in our model (by speculators we mean chartists) .

1) The Tobin tax, when seen from the point of view of the agent engaged in international capital movements, is a tax on the relevant foreign exchange transactions, but can be translated into an equivalent tax on interest income. It is, in fact, equivalent[19] either to a tax on foreign interest income at a rate which is an increasing function of the Tobin tax rate θ, or to a negative tax (i.e., a positive subsidy) on domestic interest income at a rate which is an increasing function of θ. Hence it acts by suitably modifying the interest-rate differential which enters into the CIP and UIP calculations. More precisely, let us first calculate the interest wedge brought about by the Tobin tax. For this purpose, we consider the UIP condition. The agent who owns an amount x of domestic currency can choose between

a) investing his funds at home (earning the interest rate i_h), or

b) converting them into foreign currency (and paying the Tobin tax at the rate θ) at the current spot exchange rate r, placing them abroad (earning the interest rate i_f), and converting them (principal plus interest accrued) back into domestic currency (and again paying the Tobin tax) at the end of the period considered, using the expected spot exchange rate ($\widetilde{r}$) to carry out this conversion.

The no-profit condition is

$$x\left(1+i_h\right)=\left\{\left[\frac{x(1-\theta)}{r}\left(1+i_f\right)\right]\widetilde{r}\right\}(1-\theta)=x\left(1+i_f\right)(1-\theta)^2\frac{\widetilde{r}}{r}, \quad (24)$$

unpredictability.

[18] Estimates of the parameters were found by a Gaussian estimator of the non-linear model subject to all constraints inherent in the model by using Wymer's software for the estimation of continuous time non-linear dynamic models (see Wymer, 2006).

[19] See Gandolfo and Padoan, 1992.

where the interest rates and expectations are referred to the same time horizon.

For θ sufficiently small (the values suggested by Tobin do satisfy this criterion) we have

$$(1-\theta)^2 \simeq \frac{1-\theta}{1+\theta}, \tag{25}$$

hence expression (24) becomes

$$x\,(1+i_h) = x\,(1+i_f)\,\frac{1-\theta}{1+\theta}\frac{\widetilde{r}}{r}. \tag{26}$$

If we divide both members of (26) by $x(1+i_f)(1-\theta)/(1+\theta)$, and then subtract 1 from the result, we obtain

$$\frac{(1+\theta)(1+i_h)-(1-\theta)(1+i_f)}{(1-\theta)(1+i_f)} = \frac{\widetilde{r}-r}{r},$$

whence

$$\frac{(1+\theta)i_h+2\theta}{(1-\theta)(1+i_f)} - \frac{i_f}{(1+i_f)} = \frac{\widetilde{r}-r}{r}.$$

Neglecting, as is usually done in UIP calculations, the denominator $(1+i_f)$, we finally have

$$\frac{(1+\theta)i_h+2\theta}{1-\theta} - i_f = \frac{\widetilde{r}-r}{r} \tag{27}$$

or

$$i_f = \frac{(1+\theta)i_h+2\theta}{1-\theta} - \frac{\widetilde{r}-r}{r}. \tag{28}$$

If we introduce a risk premium δ, then the condition would be

$$i_f = \frac{(1+\theta)i_h+2\theta}{1-\theta} - \frac{\widetilde{r}-r}{r} - \delta. \tag{29}$$

The r.h.s. of Eq. (28) or (29) gives the value that the foreign interest rate (referred to the appropriate time interval) has to exceed for capital outflows to be profitable[20].

Finally, it is interesting to show that the Tobin tax on foreign transactions is equivalent to a negative tax (i.e., a *subsidy*) on funds invested at home. In this latter case, in fact, we would have

$$x\,(1+i_h)\,(1+\sigma) = \left[\frac{x}{r}\,(1+i_f)\right]\widetilde{r},$$

[20] Jeffrey Frankel (1996, p. 22) gives a slightly different formula, in which the term $(\tilde{r}-r)/r$ is neglected.

where σ is the subsidy rate. Hence

$$x\,(1+i_h) = x\,(1+i_f)\,(1+\sigma)^{-1}\frac{\widetilde{r}}{r}. \tag{30}$$

If we compare (24) and (30), and let

$$\frac{1}{1+\sigma} = (1-\theta)^2,$$

we can obtain the subsidy rate equivalent to the Tobin tax, which turns out to be

$$\sigma = \frac{1-(1-\theta)^2}{(1-\theta)^2}. \tag{31}$$

For example, a Tobin tax at the rate 0.2% would correspond to a subsidy at the rate of about 0.4%.

Thus a Tobin tax influences the interest rate differential on which capital flows are based. We do not have the interest rate differential as a relevant variable in our model; however, since in our model the non-linear function g_{sc} incorporates possible transaction costs, denoting by θ a generic transaction cost we have that

$$\frac{\partial g'_{sc}}{\partial \theta} < 0.$$

Thus the introduction of a Tobin tax has the effect of decreasing the value of the adjustment speed n.

2) A Tobin tax has the effect of reducing speculators' profit. Profit (in terms of domestic currency) over a certain time interval is given by the change in the speculative stocks over that interval. The *instantaneous* change in speculative stocks is equal to the excess demand for domestic currency which, in turn, is equal to the opposite of the excess demand for foreign exchange multiplied by the exchange rate. Thus, with reference to a time interval $s_1 \vdash\!\dashv s_2, s_2 > s_1$, profit is given by

$$P = -\int_{s_1}^{s_2} E_{sc}(t)r(t)dt.$$

Equivalently, profit can be defined as the excess of the sums collected by speculators over the sums paid out by them over a given time interval, all measured in domestic currency. The sums collected in each instant are given by the amount of foreign exchange supplied, $S(t)$, multiplied by the exchange rate, and the sums paid out in each instant are given by the amount of foreign exchange demanded, $D(t)$, multiplied by the exchange rate. With reference to a given time interval we have

$$P = \int_{s_1}^{s_2} O_{sc}(t)r(t) - \int_{s_1}^{s_2} D_{sc}(t)r(t) = -\int_{s_1}^{s_2} E_{sc}(t)r(t)dt.$$

Given the definition of $E_{sc}(t)$ in our model, we have[21]

$$P = -\int_{s_1}^{s_2} [nb_1\dot{r}(t) + nb_2\dot{r}(t)]r(t)dt.$$

It is easy to see that a decrease in n (caused by a Tobin tax) would determine a decrease in P. In fact,

$$\frac{\partial P}{\partial n} = -\int_{s_1}^{s_2} [b_1\dot{r}(t) + b_2\dot{r}(t)]r(t)dt > 0,$$

since we assume that $P > 0$.Thus the qualitative effect of a change in n on P is completely determined, without having to calculate the integrals.

Let us now analyse the effects of a change in n on the model's parameters. We have

$$\begin{array}{lll}
\frac{\partial a_2}{\partial n} = & \frac{\partial}{\partial n}(\frac{m-\alpha}{nb_2}) & = -\frac{m-\alpha}{n^2b_2} < 0, \\
\frac{\partial a_3}{\partial n} = & \frac{\partial}{\partial n}(-\frac{\beta}{nb_2}) & = \frac{\beta}{n^2b_2} > 0, \\
\frac{\partial a_4}{\partial n} = & \frac{\partial}{\partial n}(\frac{g}{nb_2}) & = -\frac{g}{n^2b_2} > 0, \\
\frac{\partial a_5}{\partial n} = & \frac{\partial}{\partial n}(-\frac{m}{nb_2}) & = \frac{m}{n^2b_2} > 0.
\end{array}$$

Recall that we have been able to pinpoint the sign of g (which was uncertain) through the estimation of a_4.

Given this, we have that a *decrease* in n causes:
- an increase in a_2,
- a decrease in a_3,
- a decrease in a_4,
- a decrease in a_5.

Thus we can simulate the introduction of a Tobin tax by appropriately changing the values of the parameters. Numerical simulations will be the subject of another paper. However, a few theoretical results are given in the conclusion.

[21] To ensure that profit is not mereley notional, namely not actually realised, it is necessary to consider an interval over which the speculative stocks of foreign exchange are the same at the beginning and at the end. Thus s_1, s_2 must be such that

$$\int_{s_1}^{s_2} E_{sc}(t)dt = 0.$$

4 Conclusion

The introduction of a Tobin tax causes a decrease in speculators' profit. It also makes the model less liable to chaotic motions. In fact, in our previous paper (Federici and Gandolfo, 2011, Sect. 5) we found that when a_2 is set close to zero, the model becomes unstable and chaotic motions arise. Thus the decrease in n due to a Tobin tax, by causing an increase in a_2 as shown above, is a stabilizing factor.

5 References

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