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TURING'S ECONOMICS
A BIRTH CENTENNIAL *HOMAGE*[♣]

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> *"That the machine is digital however has more subtle significance. It means firstly that numbers are represented by sequences of digits which can be as long as one wishes. One can therefore work to* ***any desired degree of accuracy****. ... A second advantage of digital computing machines is that they are not restricted in their applications to any particular type of problem. ... With digital machines .... it is almost literally true that they are able to tackle* ***any computing problem****."*
> **Alan Turing** (1947; 1986), p. 106; emphasis added.

[♣]An earlier, much briefer, version was prepared as an invited contribution to the **OECD 'blog'** (OECD Insights), to commemorate the *Alan Turing Birth Centennial*. This *homage* is to one of 20th century's most imaginative scientists, whose works have influenced the way I have approached formalism in economic theory, behavioural economics and economic dynamics. I am deeply indebted, firstly, to Professor Barry Cooper and Dr. Patrick Love for inspiring me to embark on varieties of homages to this great human being, purest of pure scientists and one who never 'sullied' his intellectual integrity to 'feather his own nest'. Secondly, to Professor Alberto Quadrio Curzio for his latest manifestation of interest in, and encouragement of, my arcane research scribbles. I am, as always, deeply indebted to my *ASSRU* colleagues, Stefano Zambelli, Ying-Fang Kao and Ragu Ragupathy for invaluable intellectual and personal support and sustenance. None of the above worthies are responsible for any of the remaining infelicities.

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Abstract

In this paper, in homage to Alan Turing's birth centennial, I try to develop what may be called Turing's Economics. I characterize the contents of such an 'economics' in terms of the conceptual and mathematical tools developed by Alan Turing. It is shown, in more and less detail, how these concepts and tools could be used in core areas of economic theory to raise fundamental queries on claims of computability – and answer them precisely. The conclusion is that the field of Turing's Economics would – should – contribute to a reorientation of economics in the direction of serious considerations of mathematical epistemology.

Key words: Computability, Problem Solving, Undecidability, Epistemology, Computational economics

JEL Codes: B41, C63, C65, C68, D58

§1. An Intellectual Setting for *Turing's Economics*

> "We might say that the clock enables us to introduce a *discreteness* into time, so that *time* for some purposes can be regarded as *a succession of instants* instead of a continuous flow."
> Turing (1947; 1986), p. 111; italics added.

The orthodox frontier of macroeconomic theory is called *Recursive* Macroeconomics (Ljungqvist & Sargent, 2004), by which is meant a dependence of the economic theory characterizing the subject on the so-called recursive structure of dynamic programming, (Kalman) filtering and Markov Decision processes. They have nothing whatsoever to do with formal recursion theory – i.e., computability theory – and their computable content is dubious, to say the least (Velupillai, 2008).

On the other hand, in a recent Special Issue of **Economia Politica** (Quadrio Curzio, 2011), celebrating the 60th anniversary of the construction, by A.W.H. Phillips, of an electro-mechanical-hydraulic analogue computing machine to study the Keynesian macroeconomic dynamics, there were distinguished contributions arguing for the advantages of computing in *continuous*, yet transparent, mode. The mathematical foundations of such analogue computing devices are, again, not underpinned in computability theory[1].

In this paper, in celebrating the birth centennial of the founding father of computability theory, Alan Turing, an attempt is made to suggest a recursion theoretical framework for aspects of economic theory which, if systematically developed, should lead to the creation of something which can be called *Turing's Economics*. Only a skeletal apparatus will be outlined in this paper, although particular aspects and elements have been developed in seriously theoretical and empirical (i.e., *simulational*) ways over the past quarter of a century (Velupillai, et. al., 2011).

Alan Turing's birth centennial, on 23rd June, 2012[2], has been the occasion for a remarkable number of retrospectives on his fundamental and pioneering contributions to the established pure and applied sciences, on the one hand, and for the prescience with which he contributed to the philosophy, epistemology and methodology of yet to be born natural, social and

[1] Although it is easy to show that analogue devices of this sort cannot violate or surpass the limits of computation established in computability theory (cf. Velupillai's contribution to Quadrio Curzio, *op.cit.*).

[2] Strangely, in his personal memoir, *My Brother Alan*, John Turing states: 'My brother Alan was born on 21 June 1912 in a London Nursing Home' (p.145, John Turing, 2012).

cognitive sciences, too. In the latter group, computer science, artificial intelligence, computable economics and varieties of computational paradigms in physics, chemistry, biology and neurophysiology shine and reflect Turing's remarkable insights most significantly. He is also, together with Alonzo Church, Emil Post, Stephen Cole Kleene and a few others – in which list once could, but does not have to include, also Thoralf Skolem and Kurt Gödel – is regarded as the founding father of recursion theory, with important contributions to both proof theory and model theory.

It is little realised that what I call the *Five Turing Classics – On Computable Numbers* (Turing 1936-7), *Systems of Logic* (Turing, 1939), *Computing Machinery and Intelligence* (Turing, 1950), *The Chemical Basis of Morphogenesis* (1952) and *Solvable and Unsolvable Problems* (1954) – should be read together to understand why there can be something called *Turing's Economics*[3].

A comparison of Turing's classic formulation of *Solvable and Unsolvable Problems* and Simon's variation on that theme, as *Human Problem Solving* (Newell & Simon, 1972), would show that the *human problem solver* in the world of Simon needs to be defined – as Simon did - in the same way Turing's approach to *Solvable and Unsolvable Problems* was built on the foundations he had established in his classic of 1936-37.

At a deeper epistemological level, I have come to characterize the distinction between orthodox economic theory and *Turing's Economics* in terms of the last sentence of the last published paper by Alan Turing (1954, p. 23; italics added):

> "These, and some other results of mathematical logic may be regarded as going some way towards a demonstration, within mathematics itself, of the *inadequacy of 'reason' unsupported by common sense*."

This is nothing other than the obverse of what Simon advocated from almost the end of Turing's life to the end of his own[4]:

> "Both logicians and psychologists agree nowadays that logic is not to be confused with *human thinking*. For the logician, inference has objective, formal standards of validity that can exist only in Plato's heaven of ideas and not in human heads. For the

[3] The five contributions came in two clusters, the first two in 1936-7 & 1938/9; the last three in the fertile last four years of his tragically brief life.

[4] Reading 'logical' as 'reason' and 'human thinking' as 'common sense' the duality advocated by Turing and Simon become clear.

psychologist, *human thinking* frequently is not rigorous or correct, does not follow the path of step-by-step deduction – in short, is *not* usually '*logical*'."
Simon, 1991, p. 192; italics added.

We – at *ASSRU*[5] – characterize every kind of orthodox economic theory, including orthodox behavioural economics[6], advocating the *adequacy* of 'reason' unsupported by common sense; contrariwise, in Turing's economics we take seriously what we now refer to as the *Simon-Turing Precept*: 'the inadequacy of reason unsupported by common sense'.

At another frontier of research in many of what are fashionably referred to as 'the sciences of complexity', some references to Turing's classic on *The Chemical Basis of Morphogenesis* is becoming routine, even in varieties of computational economics[7] exercises, especially when concepts such as 'emergence' are invoked. Just as he had done in the case of *Solvable and Unsolvable Problems*, mulling over the nature and structure of combinatorially complex games such as Chess and GO, before interpreting the solvability of such games in terms of the mathematics of his 1936-7 classic on *Computable Numbers*, the contents of D'Arcy Thompson's classic, *On Growth and Form* (D'Arcy Thompson, [1917], 1942), preoccupied his fertile mind for over a decade and a half before the *Morphogenesis* classic came to fruition (see Hodges, 1983, pp. 207-8)[8].

It is now increasingly realized that the notion of 'emergence' originates in the works of the *British Emergentists*, from John Stuart Mill to C. Lloyd Morgan, in the half-century

[5] See http://www.assru.economia.unitn.it/, where Turing's role in our research – quite apart from the fact that he, together with Brouwer, Keynes, Sraffa, Goodwin and Simon are considered our intellectual 'patrons' – is more than evident in the various activities and publications we have sponsored and in the roles we have played in international events.
[6] Which I now refer to as *Modern Behavioural Economics*, to contrast it with the *Computably* underpinned *Classical Behavioural Economics* of Herbert Simon (Kao & Velupillai, 2012).
[7] Which has to be carefully distinguished from *Computable* – or, *Turing's – Economics*.
[8]It is a remarkable fact that Thom's classic on Structural Stability and Morphogenesis (Thom. 1975), equally inspired **by On Growth and Form**, and devoted to a dynamical systems approach to morphogenesis, as in Turing (1952), makes no reference whatsoever to the latter classic! I may be excused for adding one small personal note here. Around 1980, my Cambridge *maestro*, Richard Goodwin, also passionate about the dynamical systems interpretation of morphogenesis and dedicated to an interpretation of Schumpeterian 'creative destruction' in terms of it, asked me which of his books I wanted. With some embarrassment I indicated that I would wish to have his copies of Schumpeter's classics and the original edition of **On Growth and Form**. He warmly agreed to my 'request', and marked his copies of these books with a note that they were to go to me, whenever time's tenancy on his life ran out! Alas, his home was 'burgled' shortly after his death and I never had the pleasure of inheriting his classics by Schumpeter and D'Arcy Thompson!

straddling the last quarter of the 19th and the first quarter of the 20th century. However, a premature Obituary of *British Emergentism* was proclaimed on the basis of a rare, rash, claim by Dirac:

> "The underlying physical laws necessary for the mathematical theory of a large part of physics and *the whole of chemistry are thus completely known*, and the difficulty is only that the exact application of these laws leads to equations much too complicated to be soluble. It therefore becomes desirable that approximate practical methods of applying quantum mechanics should be developed, which can lead to an explanation of the main features of complex atomic systems *without too much computation*."
> Dirac (1929), p. 714; italics added.

Contrast this with Turing's wonderfully laconic, yet eminently sensible precept (Turing, 1954, p. 9; italics added):

> "No mathematical method can be *useful* for *any* problem *if it involves much calculation*."

Turing's remarkably original work on *The Chemical Basis of Morphogenesis* was neither inspired by, nor influenced by any later allegiance to the *British Emergentist*'s tradition. Perhaps Turing (and Thom, too) was simply too 'innocent'[9] to establish the source of the framework he was devising in a tradition; instead, facing and solving each problem *de novo*, and devising his own vision of, and for, it (as he did with the construction of the abstract notion of a Turing Machine to solve a metamathematical query, i.e., the *entscheidungsproblem*). It is, I think, this attitude, coupled to supreme confidence in his own intellectual powers, that was elegantly summarized by Hodges (2008, p. 4):

> "It was typical for him …. To seek to outdo Bell Telephone Laboratories with his single brain, and to build a better system with his own hands."

On the other hand, the structure of the experimental framework Turing chose to construct was uncannily similar to the one devised by Fermi, Pasta and Ulam, (1955), although with different purposes in mind. But there was – and there remains – a deeper affinity in that the violation of the equipartition of energy principle that was observed in the Fermi-Pasta-Ulam simulation and the symmetry-breaking that is intrinsic to the dynamical system behaviour of Turing's system of reaction-diffusion equations.

Turing's aim was to devise a mechanism by which a spatially homogeneous distribution of chemicals – i.e., formless or patternless structure - could give rise to form or patterns via

[9]In the fertile sense in which one thinks a child is innocent.

what has come to be called a *Turing Bifurcation*. A reaction-diffusion mechanism formalised as a (linear) dynamical system and subject to what I have referred to, in other writings, as *the linear mouse theory of self-organisation*[10].

In this same vein, it is most satisfying to note the role the *Turing Bifurcation* played in the development of the *Brusselator* and the work of the 1977 Chemistry Nobel Prize winner, Ilya Prigogine (cf. Nicoils and Prigogine, 1977) on self-organisation in non-equilibrium systems.

Those seriously interested in the nonlinear, endogenous, theory of the business cycle, know very well that the *Turing Bifurcations* are at least as relevant as the *Hopf Bifurcation*, in modeling the 'emergence' and persistence of unstable dynamics, in aggregative economic dynamics.

Turing's Economics straddles the micro-macro divide in a way that makes the notion of microfoundations of macroeconomics thoroughly irrelevant; more importantly, it is also a way of circumventing the excessive claims of reductionists in economics, and their obverse! This paradox would have, I conjecture, provided much amusement to the mischievously innocent child that Turing was, all his life.

The paper is structured as follows. The next section is a somewhat 'lighthearted' discussion on Turing *and* Economists. In §3 I try to summarise, in a concise fashion, the main results of

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[10] In typically playful fashion, he summarised the mathematical mechanism he sought (Turing, 1952, pp. 43-4; italics added):

> "Unstable equilibrium is not … a condition which occurs very naturally. .. Since systems tend to leave unstable equilibria they cannot often be in them. *Such equilibria can, however, occur naturally through a stable equilibrium changing into an unstable one*. For example, if a rod is hanging from a point a little above its centre of gravity it will be in stable equilibrium. If, however, a mouse climbs up the rod the equilibrium eventually becomes unstable and the rod starts to swing. … The system which was originally discussed … might be supposed to correspond to the mouse *somehow reaching the top* of the pendulum without disaster, perhaps by falling vertically on to it."

Contrast this thoroughly unfazed interpretation of 'unstable equilibrium' with even the otherwise enlightened Hicksian understanding of this (Hicks, 1949, p. 108; italics added):

> "But mathematical *instability* does *not* in itself *elucidate fluctuation*. *A mathematically unstable system does not fluctuate*; it just breaks down. The unstable position is one in which it will *not* tend to remain. That is *all* that the condition of mathematical instability tells us. But, on being barred from that position, what will it do? What path will it follow? *Mere knowledge* of the unstable position does not tell us."

Even the Gods nod! Turing and all nonlinear, endogenous, mathematical macrodynamists would, with good reason, dispute every claim in this strange indictment of 'mathematical instability' by the doyen of 20th century economic theory (cf. Ragupathy & Velupillai, 2012).

Turing's Economics, with some discussion of the kind of mathematical framework required to achieve these results in a consistent and rigorous mode. Finally, in the concluding §4, I speculate – though not frivolously – on the theme of Whither Turing's Economics?, on the basis of what has been achieved, mostly against the teeth of ignorant orthodox theoretical objections.

§2. Turing and Economists

> "If we hurry, we can catch up to Turing on the path he pointed out to us so many years ago."
> Simon, 1996, p.81.

Herbert Simon, as one of the acknowledged founding fathers of *computational cognitive science* was deeply indebted to Turing in the way he tried to fashion what I have called *Computable Economics* (Velupillai, 2000)[11]. Simon was 'on the path that [Turing] pointed to us so many years ago' (Kao & Velupillai, *op.cit*.), but there is no recorded evidence that Simon ever met, or corresponded with, Turing. The tragic end of Turing's life, just as Simon was entering his own 'computable' and computational complexity theoretic' phase of his astonishingly fertile intellectual life[12], is, surely the only reason for this lacuna. It has been my intellectual mission, for at least thirty years, first to learn to take this 'path', and then to teach others the excitement and fertility, for economic research, of taking the path Turing 'pointed out to us so many years ago'.

To the best of my knowledge the only economist who took this path – of course with the noble – and *Nobel* – exception of Herbert Simon, albeit almost simultaneously with my journey along it[13], was Alain Lewis (cf. Velupillai, et. al., chapter 1). But we came to take that path fully over thirty years after the death of Turing and Simon's initial forays into *Turing's Economics*.

[11] I could as well have called it *Turing's Economics*.

[12] 'The most important years of my life as a scientist were 1955 and 1956, when the maze [that was my intellectual life] branched in a most unexpected way.' (Simon, 1991, p. 189).

[13] My personal **Turing Number (*pace* Erdös Number!)** may well be 2.5! One of my first papers in what I now wish to call Turing's Economics was co-authored with John Westcott (& Berc Rustem) and published in **Automatica**, in 1978 (cf., Rustem, et.al., 1978). John Westcott and Alan Turing were members of the Ratio Club (cf., Husbands, et. al.,2008, ch. 6 and, in particular, p. 124, in which Turing and Westcott appear together in the group photograph taken at *Peterhouse*, Cambridge). In 1980, I succeeded my mentor, Richard Goodwin, as the Director of Studies in Economics at *Peterhouse*! There is clear evidence (Husbands, et.al., *loc.cit*, p. 116) that Turing and Westcott spoke at the same *Ratio Club* meeting on 21 April, 1950. John Westcott is, happily, still alive and well and I have resumed my correspondence with him after an absence of 35 years and I hope to see and talk with him about his memories of Turing on my next visit to England.

During his tragically shortened lifetime Turing was acquainted with three distinguished economists, all fellow 'Kingsmen'[14]: A.C. Pigou, Maynard Keynes and David Champernowne. However, there is no evidence whatsoever that any of Turing's fundamental contributions played the slightest role, or had any influence at all, on the outstanding economic theories and policy proposals developed and advocated by this trio of remarkable King's economists.

Pigou and Keynes, as two of the (four – the other two were Philip Hall and the Provost of King's, John Sheppard) readers of his fellowship dissertation, submitted to King's College in November 1934 (Turing was only 22 years old) '*On the Gaussian Error Function*', enthusiastically supported his election. He was duly elected a Fellow of King's College on March 16, 1935 (cf. Hodges, 1983, p. 88; Zabell, 1995).

As Zabell (*loc. cit*, p. 483), perceptively and with clear appreciation and admiration, observes:

> "[Turing's] name is not usually thought of in connection with either probability or statistics. One of the basic tools in both of these subjects is the use of the normal or Gaussian distribution as an approximation, one basic result being the Lindeberg-Feller central limit theorem taught in first-year graduate courses in mathematical probability. No one associates Turing with the central limit theorem, but in 1934 Turing, while still an undergraduate, rediscovered a version of Lindeberg's 1922 theorem and much of the Feller-Lévy converse to it (then unpublished."

Turing's friendship with Pigou blossomed in many ways, most of which are touchingly described and narrated in Sara Turing's affectionate biography of her son: Alan M. Turing (1959, 2012). There is, however, exactly one incident, in Turing's friendship with Pigou, that resulted in a tangential contribution to economics by the former, as a result of a query posed by the latter:

> "Once I [Pigou] remember I put to him [Turing] a matter in which I was having a discussion with an American economist, the full solution of which required some [to Pigou] rather difficult mathematics. He found that I was wrong and the American right, but the mathematical argument with which the American purported to support his case was wrong. He himself worked out what I presume was a valid argument, but would not let it be published because he said, '*whatever it might be as economics, as mathematics it was not interesting*.'"
>
> Sara Turing, *op.cit*, pp. 86-7; italics added.

[14] In other words, King's College Cambridge.

How much of the mathematics of modern so-called mathematical economics would have seemed ‘interesting’ for Turing is a question worth asking!

But it was his lifelong friendship with David Champernowne that leaves an unresolvable puzzle: there is absolutely no evidence whatsoever, in Champernowne’s many fundamental, path-breaking, contributions to economic theory, mathematics, applied economics or even to ‘Uncertainty and Estimation’ theory (Champernowne, 1969), of Turing’s influence – whether via computability theory or anything else to which Turing contributed.

However, there is one marginal sense in which there may be a legacy, via Herbert Simon, for whom Chess was the paradigmatic example where human problem solvers were studied to extract lessons on boundedly rational behaviour of agents satisficing. According to Donald Michie (Sara Turing, *op.cit*, p. 96; but see also Hodges, 1983, p. 388):

> “Alan told me that he and David Champernowne had constructed a machine to play chess, in the sense of a complete specification on paper for such a machine. … During a stay in Cambridge, Shaun Wylie and I constructed a rival ‘paper machine’ which we christened Machiavelli, from our two names, Michie-Wylie. On behalf of Machiavelli we then issued a challenge to Turochamp (our name for the Turing-Champernowne machine, the game to be played by correspondence.”

The best ‘estimate’ of a date for the construction – or, at least, the conception of the *Turochamp* suggests it predated Shannon’s contribution (Shannon, 1950), which initially influenced Simon’s later lifelong adherence to Chess as a paradigmatic example to study human problem solvers in action, to the same topic by about a decade.

On the other hand, the reverse influence, of Champernowne on Turing, may not have been inconsiderable – but on the philosophy and methodology of mathematics. This is eloquently and convincingly argued in Hodges (2008) on the basis of the ‘friendly rivalry’ induced in Turing, as a result of Champernowne’s remarkable ‘undergraduate’ contribution to the explicit construction of a number normal to the base 10 (Champernowne, 1933)[15].

One can trace the unfulfilled promise of developing a theory of *constructive procedures for defining real numbers*, announced in his classic of 1936 (Turing, 1936-7), to the inspiration from Champernowne (1933).

[15] Champernowne’s explicit construction was the remarkable number: 0.123456789101112 …!

Apart from David Champernowne, one of Turing's closest friends was his contemporary, Alister Watson who, in fact, was the person who introduced him to Wittgenstein. Both Champernowne and Watson were, in turn, Piero Sraffa's friends, perhaps in different ways. Champernowne's lucid 'Note' on von Neumann (Champernowne, 1945-6), which introduced the difficult – for the times – mathematics underpinning it in economic contexts and terms, paid handsome acknowledgements to Sraffa's 'instruction in subjects discussed in [the] article' (*loc. cit*., p. 10). Watson, on the other hand, has the 'obverse' boot on – in that he, together with Besicovitch and Ramsey, are thanked by Sraffa for 'invaluable mathematical help over many years' (Sraffa, 1960, p. vi). Moreover, in Sraffa's early years he, too, was a member of King's College. Finally, there is also the famous friendship between Sraffa and Wittgenstein.

However, whether all this 'circumstantial evidence', leading to some detectable relationship between Turing and Sraffa, can be substantiated only by serious archival explorations in the Turing papers at King's College and the Sraffa papers at Trinity college.

My own interest in this possibility has much to do with the nature of the way Sraffa used mathematical formalisms and the mode of formal demonstrations of propositions in his book – i.e., methods of proof. I have always maintained that Sraffa's methods of proof are constructive, at least in his famous book (Velupillai, 2008a). Ramsey, Wittgenstein, Watson and Champernowne were, all of them, well acquainted with the controversies in the foundations and the philosophy of mathematics, and all of them, in one way or another, adhered to, or seriously sympathized with, the intuitionists and their brand of constructivism.

Surely, it is not too far-fetched a conjecture that some, at least, of these issues 'rubbed off' on the mighty intellectual force that was Sraffa!

§3. Turing's Economics - *Achievements*

> "The next step in analysis[16], I would conjecture, is a more consistent assumption of *computability* in the formulation of economic hypotheses. This is likely to have its own difficulties because, of course, not everything is *computable*, and there will be in this sense an inherently unpredictable element in rational behavior."
> Arrow, 1987, p. S398; italics added

[16] Clearly, the context implies that this refers to `the next step in economic analysis'.

After reading my Arne Ryde Lectures (Velupillai, 2000), Herbert Simon wrote me (on 25 May, 2000; italics added) as follows:

> " As the book makes clear, my own journey through bounded rationality has taken a somewhat different path. Let me put it this way. There are many levels of complexity in problems, and corresponding boundaries between them. *Turing computability* is an outer boundary, and as you show, any theory that requires more power than that surely is irrelevant to any useful definition of human rationality. ...."

Simon was referring to my two results in the above book (Chapter 3), derived by formalizing orthodox rational choice theory – that which Simon referred to as *Olympian rationality* (Simon, 1983, p. 19) – as the computing activities of a Turing Machine:

> **Theorem 1**:
>
> There is no *effective procedure* to generate preference orderings.
>
> **Theorem 2**:
>
> Given a class of choice functions that do generate preference orderings (pick out *the set of maximal alternatives*) for any agent, there is no *effective procedure* to *decide* (algorithmically) whether or not any arbitrary choice function is a member of the given class.

Their essential implication is that the optimization activity of *Olympian rationality* is algorithmically meaningless.

On 21st July, 1986, Arrow wrote as follows to Alain Lewis (Arrow Papers, Box 23; italics added):

> "[T]he claim the *excess demands are not computable* is a much profounder question for economics that the claim that equilibria are not computable. The former challenges economic theory itself; if we assume that human beings have calculating capacities not exceeding those of Turing machines, then the *non-computability of optimal demands is a serious challenge to the theory* that individuals choose demands optimally."

That 'the excess demands are not computable' was one of the results I was able to prove, using one of Turing's enduring results – the *Unsolvability of the Halting Problem for Turing Machines*.

To prove that the excess demand functions are not computable, my strategy was to 'take off' from one half of the celebrated Uzawa Equivalence Theorem (Uzawa, 1962), which is, by the way, the basis for Scarf's pioneering work on computable general equilibrium models. This half of the theorem shows that the Walrasian Equilibrium Existence Theorem (WEET) implies the Brouwer fix point theorem and the finesse in the proof is to show the feasibility of devising a continuous excess demand function, $\boldsymbol{X}(\boldsymbol{p})$, satisfying Walras' Law (and homogeneity), from an arbitrary continuous function, say $\boldsymbol{f}(.)$: $\boldsymbol{S} \rightarrow \boldsymbol{S}$, such that the equilibrium price vector implied by $\boldsymbol{X}(\boldsymbol{p})$ is also the fix point for $\boldsymbol{f}(.)$, from which it is 'constructed'. The key step in proceeding from a given, arbitrary, $\boldsymbol{f}(.)$: $\boldsymbol{S} \rightarrow \boldsymbol{S}$ to an excess demand function $\boldsymbol{X}(\boldsymbol{p})$ is the definition of an appropriate scalar:

$$\mu(p) = \frac{\sum_{i=1}^{n} p_i f_i[\frac{p}{\lambda(p)}]}{\sum_{i=1}^{n} p_i^2} = \frac{p.f(p)}{|p|^2} \tag{1}$$

Where:

$$\lambda(p) = \sum_{i=1}^{n} p_i \tag{2}$$

From (1) and (2), the following excess demand function, $X(p)$, is defined:

$$x_i(p) = f_i(\frac{p}{\lambda(p)}) - p_i \mu(p) \tag{3}$$

i.e.,

$$X(p) = f(p) - \mu(p)p \tag{4}$$

It is simple to show that (3) [or (4)] satisfy:

(i). $\boldsymbol{X}(\boldsymbol{p})$ is continuous for all prices, $\boldsymbol{p} \in \boldsymbol{S}$

(ii). $\boldsymbol{X}(\boldsymbol{p})$ is homogeneous of degree 0;

(iii). $\boldsymbol{p} \bullet \boldsymbol{X}(\boldsymbol{p}) = 0$, $\forall \boldsymbol{p} \in S$, i.e., Walras' Law holds:

$$\sum \boldsymbol{p_i x_i}\,(\boldsymbol{p}) = \boldsymbol{0}, \qquad \forall p \in S \ \& \ \forall i = 1.....n \tag{5}$$

Hence, $\exists \boldsymbol{p}^*$ s.t., $\boldsymbol{X}(\boldsymbol{p}^*) \leq 0$ (with equality unless $\boldsymbol{p}^*=0$). Elementary logic and economics then imply that $\boldsymbol{f}(\boldsymbol{p}^*)=\boldsymbol{p}^*$. I claim that the procedure that leads to the definition of (3) [or, equivalently, (4)] to determine $\boldsymbol{p}^*$ is provably *undecidable*. In other words, the crucial scalar in (1) cannot be defined recursion theoretically to *effectivize* a sequence of projections that would ensure convergence to the equilibrium price vector.

Theorem 3:

$\boldsymbol{X}(\boldsymbol{p}^*)$, as defined in (3) [or (4)] above is undecidable; i.e., cannot be determined algorithmically.

Proof:

Suppose, contrariwise, there is an algorithm which, given an arbitrary $\boldsymbol{f}(.)$: $\boldsymbol{S}\rightarrow\boldsymbol{S}$, determines $\boldsymbol{X}(\boldsymbol{p}^*)$. This means, therefore, in view of (i)-(iii) above, that the given algorithm determines the equilibrium $\boldsymbol{p}^*$ implied by WEET. In other words, given the arbitrary initial conditions $\boldsymbol{p} \in \boldsymbol{S}$ and $\boldsymbol{f}(.)$: $\boldsymbol{S}\rightarrow\boldsymbol{S}$, the assumption of the existence of an algorithm to determine $\boldsymbol{X}(\boldsymbol{p}^*)$ implies that its *halting configurations* are *decidable*. But this violates the ***undecidability of the Halting Problem for Turing Machines***. Hence, the assumption that there exists an algorithm to determine - i.e., to construct – $\boldsymbol{X}(\boldsymbol{p}^*)$ is untenable.

Almost sixty years ago, Patinkin (1956, p11; 1965, p. 7), made a deceptively simple claim on the feasibility of 'constructing' excess demand functions:

> "Indeed, we can consider the individual – with his given indifference map and initial endowment ***P*** – to be a 'utility computor'[17] into whom we 'feed' a sequence of market prices and from whom we obtain a corresponding sequence of 'solutions' in the form of specified optimum positions. In this way we can conceptually generate the individual's excess-demand curve for, say, ***X***; this shows the excess amounts of ***X*** he demands at the various prices."

The implications of the above three theorems make this kind of *thought-experiment*[18] utterly meaningless – long before any of the fashionable invoking of the so-called Sonnenschein-Debreu-Mantel theorem(s) even make their appearance on the scene. Patinkin's thought-experiment is no different from an engineer claiming to build a perpetual-motion machine, violating the second law of thermodynamics, which is, by the way, a 'law', not a theorem, just as much as the *Undecidability of the Halting Problem* is underpinned by a 'thesis' not a 'theorem' – the Church-Turing Theorem.

Exactly analogous results, on the production side of the 'orthodox' coin, are derivable, as has been shown in Luna (2004), Velupillai (2010) and Zambelli (2004, 2005[19]). In addition to

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[17] Note: 'computor' – not 'computer'! But, of course, Turing's classic (1936) was about the human computer, which is what Patinkin intends with his 'computor'.

[18] Incidentally, Patinkin does not even observe the elementary strictures of a 'function for thought experiments' (see, Kuhn, 1964; 1977). It is, however, no worse than Leijonhufvud's equally silly invoking of the analogy of *Maxwell's Demon* to suggest the 'function' of the *Walrasian Auctioneer* (Leijonhufvud, 1981, p. 15).

[19] I would like to point out that this important work by Zambelli was published in a book dedicated to the memory of Alan Turing. The dedication in full is: 'The authors of this volume dedicate their work to the noble memory of Alan Turing for the purity of intellectual spirit he displayed in his seminal writings.' The book itself was meant to commemorate the 70th anniversary of Turing's election as a Fellow of King's College, Cambridge

encapsulating production *processes* as the computing behavior of Turing Machines (Luna, *loc.cit*., & Velupillai, *loc.cit*.), there is Zambelli's pioneering harnessing of the *Busy Beaver* in studying the evolution of 'ideas' in the sense defined by Romer (1993). Moreover, both Luna and Zambelli, in these two respective studies demonstrate, theoretically and simulationally, how the Turing Machine Metaphor eschews need for *ad hoc* reliance on any kind of arbitrary, exogenous, probability or stochastic structure to generate technological processes of innovation.

Learning rational expectations equilibria, in an overlapping generations model (Velupillai, 2005) and learning in traditional macroeconomic policy contexts, transformed into a computability structure (Luna, 2010), have also led to rigorously formal and effectivisable learning rules. This is to be contrasted with thoroughly non-effective rational expectations learning rules, appealing to uncomputable fix-point theorems (Sargent, 1993), in standard theory.

For example, the following four classic computability theorems are used to prove the *uncomputability* of rational expectations equilibria in orthodox frameworks and to construct *computable rational expectations equilibria*, in what I would now call macroeconomics in the Turing mode, respectively.

Rice's Theorem:
Let ***C*** be a class of *partial recursive functions*. Then ***C*** is *not* recursive unless it is the empty set, or the set of all partial recursive functions.

(Recursion Theoretic) Fix-Point Theorem:
Suppose that $\Phi:\mathcal{F}_m \rightarrow \mathcal{F}_n$ is a recursive operator (or a recursive program **Þ**). Then there is a partial function $\boldsymbol{f_\phi}$ that is the *least fixed point* of $\boldsymbol{\Phi}$:

Remark:
If, in addition to being partial, f_ϕ is also total, then it is the *unique* least fixed point.

Recursion Theorem
Let ***T*** be a Turing Machine that computes a function:

– also the *Alma Mater* of this author – for his dissertation on the *Central Limit Theorem of Probability* (see above, §2).

$$t : \Sigma^* \times \Sigma^* \longrightarrow \Sigma^* \tag{6}$$

Then, there is a Turing Machine R *that computes a function*:

$$r : \Sigma^* \longrightarrow \Sigma^* \tag{7}$$

such that, $\forall \omega$:

$$r(\omega) = t(\langle R \rangle, \omega) \tag{8}$$

where,
<***R***>: denotes the encoding of the Turing Machine into its standard representation as a bit string;
and the $*$(star) operator denotes its standard role as a unary operator defined as:
$\boldsymbol{A}^* = \{x_1, x_2, ..., x_k | \; k \geq 0, \forall x_i \in \boldsymbol{A}\}$

The idea behind the recursion theorem is to formalize the activity of a Turing Machine that can obtain *its own description* and, then, compute with it. This theorem is essential, too, for formalizing, recursion theoretically, a model of endogenous growth (Velupillai, 2010) and to determine and learn, computably and constructively, rational expectations equilibria (Velupillai, 2004). The fix point theorem and the recursion theorem are also indispensable in the computable formalization of policy ineffectiveness postulates, time inconsistency and credibility in the theory of macroeconomic policy. Even more than in microeconomics, where topological fix point theorems have been indispensable in the formalizations underpinning existence proofs, the role of the above fix point theorem and the related recursion theorem are absolutely fundamental in Turing's Economics – where, of course, the Turing Machine is the basic building block.

Anyone who is able to formalize these theorems, corollaries and conjectures and work with them, would have mastered some of the key elements that form the core of the necessary mathematics of Turing's Economics.

§4. Whither Turing's Economics?

> "I am sure that you will be able to interpret these very sketchy remarks, and I hope you will find reflected in them my pleasure in your book[20]. While I am fighting on a somewhat different front, I find it greatly comforting that these outer ramparts of *Turing computability* are strongly manned, greatly cushioning the assault on the inner lines of empirical computability."
>
> Herbert Simon's *Letter* to Velupillai, 25 May, 2000; italics added.

[20] This is a reference to Velupillai (2000).

Manning the ‘outer ramparts of Turing computability’, in modelling core areas of economic theory is a challenging task, especially in an era when so many diverse claims are made for computation in orthodox economic theoretical frameworks. Formally demonstrating that none of these claims are tenable, from the point of view of Turing computability (or Brouwerian Constructivism), has been part of my research program on Turing’s Economics for the past thirty years (cf. for example, Velupillai, 2013).

The exercise showing the untenability of the computability (and constructivity) claims of varieties of computational economics may seem a negative task. However, there are two necessary virtues in such an exercise: firstly, there are always *positive aspects of negative results*; secondly, clearing the air of false and untenable claims helps in understanding, and locating, reasons for such assertions, thus preventing further errors along the same lines.

The positive aspects of the negative results is easy to substantiate with one clear example: Herbert Simon’s development of (classical) behavioural economics, solidly and rigorously cased on computability and computational complexity theory (Kao & Velupillai, 2012). None of the conundrums implied by the three theorems on choice theory are relevant for a behavioural economics squarely underpinned by computability and computational complexity theory.

The second aspect can be illustrated as follows.

There are at least five frontier research fields in economics, encompassing both micro and macro aspects of economic theory, where machine computation is claimed to play crucial roles in formal modelling exercises:

1. **Computable General Equilibrium Theory** (CGE) (and its `extensions': Recursive Competitive Equilibrium (RCE) & Dynamic Stochastic General Equilibrium (DSGE) theories) **-** *The Scarf Tradition*
2. **Computable General Equilibrium *Modelling* -** The Johansen-Stone Tradition (cf., Johansen, 1960/1974, Dixon & Parmenter, 2009)
3. Varieties of **Agent based computational economics** (Tesfatsion, 2006, Epstein, 2006)[21]

[21] The word `constructive' in the title of the article by Tesfatsion (*loc.cit*) has nothing whatsoever to do with any `variety of constructive mathematics'. The claims, definitions and characterisations of constructive mathematical aspects of agent-based economics in Epstein (*op.cit*., chapter 1) are technically incorrect. In particular, the characterisation of nonconstructive existence proofs in terms of acceptance of the *tertium non datur*, implying that the contrary in the case in constructive existence proofs (ibid, pp. 11-12), is not accurate (cf., Brouwer, 1908). This is because the author has not stated the kind of *tertium non datur* unacceptable in constructive mathematics. Above all conflating computability theoretic statements, themselves imprecisely stated, with constructive ones makes the whole argument meaningless, from a mathematical point of view. Moreover, the

4. **Classical Behavioural Economics** (CBE, as distinct from MBE: *Modern Behavioural Economics*, see Kao & Velupillai, *op.cit*)
5. Computable economics/Turing's Economics

The main claim here is that there is a serious epistemological deficit in all of the approaches, but can be `discovered' only in the last two, precisely because the latter are underpinned by computability (and constructivity) theories, in their strict mathematical senses, and the former are not. Therefore, it is important to show in what sense, where and how the first three approaches fail in any computable (or constructive – i.e., algorithmic) claim.

The epistemological deficit, I have suggested (Velupillai, 2013), can be resolved by a theory of simulation, itself based on recognising the double `duality' between dynamical systems and numerical analysis, on the one hand, and that between computation and simulation, mediated by dynamical systems, both 'dualities' interpreted computably or constructively, leading to the core triad of computation, simulation and dynamics (because numerical analysis can be interpreted, equivalently, in terms of dynamical systems or computably). Hence, hopefully, paying heed to Turing's Precept: "the inadequacy of `reason' unsupported by common sense."

It is in this sense that I would seek to answer the question: *Whither Turing's Economics*? The exercise – and the research program – in developing an economic theory underpinned by computability theory means not just an alternative mathematical methodology for formalization. Such an approach runs the risk of simply hoping that an alternative mathematisation of economic theory is a panacea to the ills of current orthodoxy in economic theorizing, blinded by indiscriminate mathematical formalisations.

The answer I suggest, to the question posed as the heading for this section – Whither Turing's Economics? – is a reorientation of economic theory underpinned by strong adherence to epistemology, especially in the sense of Husserl, but more generally in terms of phenomenological epistemology. I believe this will lead to a fruitful synthesis of Husserl's phenomenology, Brouwer's constructivism and Turing's computability.

author does not seem to realise the elementary fact that there are celebrated nonconstructive proofs that do not invoke 'proof by contradiction' (eg., Gale, 1974).

No one encapsulated this outlook more directly, yet concisely, than the versatile Richard Feynman (1996, p. xiii: italics added):

> Computer science touches on a variety of deep issues. ... . It naturally encourages us to ask questions about *the limits of computability*, about *what we can and cannot know about the world around us*."

The arrogance of orthodox mathematisation of economic theory is the lack of appreciation of *the limits of mathematics*. The practitioner of Turing's economics – whether via classical behavioural economics or computable economics – has, from the outset, to come to terms with *the limits of computability* and, hence, about *what we can and cannot know about the world around us*.

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