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THE RELEVANCE OF COMPUTATION IRREDUCIBILITY AS *COMPUTATION UNIVERSALITY* IN ECONOMICS[♣]

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♣ Invited contribution to the volume, **Irreducibility and Computational Equivalence: 10 Years After Wolfram's A New Kind of Science**. I am greatly indebted to my friend, Hector Zenil, for the kind invitation to contribute to this important commemorative volume. One minor caveat should be added here. I subscribe to the entirely sensible view of Chris Moore & Stephan Mertens that `old fashioned topics, like formal languages and automata' are better left out of discussions at the frontiers of the theory of computation (Moore & Mertens, 2011, p. xvii).

Abstract

Stephen Wolfram's *A New Kind of Science* should have made a greater impact in economics - at least in its theorising and computational modes - than it seems to have. There are those who subscribe to varieties of agent-based modelling, who do refer to Wolfram's paradigms – a word I use with the utmost trepidation – whenever simulational exercises within a framework of cellular automata is invoked to make claims on complexity, emergence, holism, reduction and many such 'buzz words'. Very few of these exercises, and their practitioners, seem to be aware of the deep mathematical – and even metamathematical – underpinnings of Wolfram's innovative concepts, particularly of computational equivalence and computational irreducibility in the works of Turing and Ulam. Some threads of these foundational underpinnings are woven together to form a possible tapestry for economic theorising and modelling in computable modes.

Keywords: Computational equivalence, computational irreducibility, computation universality.

1 A Preamble on the Origins of the Visions[1]

> "From that point on [i.e., from January, 1982], Wolfram's bibiliography, his list of scientific production, goes from no cellular automata at all to 100 per cent cellular automata. *He decided that cellular automata can do anything.* From that moment on Stephen Wolfram became the Saint Paul of cellular automata."
>
> Toffoli, in Regis (1987), p. 238; italics added.

Even as we celebrate a decade of experiences and adventures with **A New Kind of Science**[2], we should bear in mind that the romantic seeds of the vision that became the core of NKS, *computational irreducibility*, were sown at a quintessential private[3] *Island in the Sun* in the Carribean, Moskito Island in January, 1982, thirty years ago - and that this is also the year we commemorate the Birth Centennial of Alan Turing, whose genius lies at the heart of Wolfram's sustained efforts at creating a new paradigm for scientific practice, particularly from an *inductive*[4] point of view.

It seems to me, as an economist who practices its formalisation in computable and dynamical systems theory modes, that the *Computational Irreducibility* vision comes against the backdrop of an all permeating *Complexity Vision* of economics. I am convinced that this is, proverbially, a case of placing the cart(s) before the horse(s): it is computability and dynamical systems theory - and their fertile interaction - that underpins the computational irreducibility vision which, in turn, provides one kind of foundation[5] for a complexity vision, particularly of economics.

Emergence, *order*, *self-organisation*, *turbulence*, *induction*, *evolution*, *criticality*, *adaptive*, *non-linear*, *non-equilibrium* are some of the words that characterise the conceptual underpinnings of the 'new' *sciences of complexity* that seem to pervade some of the frontiers in the natural, social and even the human sciences. Not since the heyday of *Cybernetics* and the more recent brief-lived ebullience of *chaos* applied to a theory of everything and by all and sundry, has a concept become so prevalent and pervasive in almost all fields, from Physics

[1] I shall assume readers of this volume will have at least a nodding acquaintance with the concepts of the *Principle of Computational Equivalence*, *Computational Irreducibility* and the related notion of *Algorithmic Incompressibility* (cf., for example, Li & Vitanyi (1997), chapter 6).

[2] Henceforth, referred to as **NKS**.

[3] Owned by Ed Fredkin who himself, with the **Fredkin-Zuse Thesis** which, put in an ultra simple way, states that **The Universe is a Computer**, espouses a vision that is embodied in **NKS**. Fredkin articulated this vision much later than he had conceptualised it in his actual adherence to working with such a metaphor as his guiding scientific, inductive, principle.

[4] I do not want to emphasise Inductive at the expense of *Abductive*, especially by contrasting the former exclusively as an alternative to Deductive. The mischief indulged in by economists, particularly those advocating blind agent-based modelling in economics and finance, claiming that their practice makes the case against formal mathematics in its deductive underpinnings, enhancing the case for a 'new mathematics' that is inductively based, shunts research towards pointless ends.

[5] I have in mind, as another kind of foundation, the philosophy and epistemology that came with the vision of the *British Emergentists*.

to Economics, from Biology to Sociology, from Computer Science to Philosophy as *Complexity* seems to have become. An entire Institution, with high-powered scientists in many of the above fields, including several Nobel Laureates from across the disciplinary boundaries as key permanent or visiting members, has come into existence with the specific purpose of promoting the *Sciences of Complexity*[6].

I have found Duncan Foley's excellent characterisation of the *objects* of study by the 'sciences of complexity' in Foley (2003), p.2 (italics added), extremely helpful in providing a base from which to approach the study of a subject that is technically demanding, conceptually multi-faceted and philosophically and epistemologically highly inhomogeneous:

> Complexity theory represents an ambitious effort to analyse the functioning of highly organized but decentralized systems composed of very large numbers of individual components. The basic processes of life, involving the chemical interactions of thousands of proteins, the living cell, which localizes and organizes these processes, the human brain in which thousands of cells interact to maintain consciousness, ecological systems arising from the interaction of thousands of species, the processes of biological evolution from which *new species emerges*, and *the capitalist economy, which arises from the interaction of millions of human individuals, each of them already a complex entity, are leading examples.*

It is one thing to observe similarities at a phenomenological and structural level. It is quite another to claim that one 'science', with its own characteristic set of methods, can encapsulate their study in a uniform way, thus providing rationale for an interdisciplinary approach to all of them. Here again, I believe the elegant attempt to go just below the surface similarities of phenomena and structure, and define the conceptual and methodological underpinnings of this new 'science' in Foley (*ibid*), is most illuminating:

> What these [highly organized but decentralized] systems share are a **potential** to configure their component parts in an astronomically large number of ways (they are *complex*), constant change in response to environmental stimulus and their own development (they are *adaptive*), a strong **tendency** to achieve recognizable, stable patterns in their configuration (they are *self-organising*), and an avoidance of stable, **self-reproducing** states (they are *non-equilibrium systems*). The task complexity science sets itself is the

[6] I am referring, of course, to the *Santa Fe Institute*, which, refreshingly, has thought it prudent to have a permanent Economics division from the outset. But, on a sceptical note, the almost untrammelled enthusiasm for a unified vision for all of the disciplines has the danger, in my opinion, of making essentially moral, human and social sciences like economics a handmaiden to the concepts and methods of the natural sciences and, in this sense, we seem to be travelling along well trodden and mistaken paths of the past. Vico's famous dictum keeps coming back to haunt my mind: *'Corsi e ricorsi ...*!

exploration of the general properties of complex, adaptive, self-organizing, non-equilibrium systems.

The *methods* of complex systems theory are highly empirical and inductive. ... A characteristic of these ... complex systems is that their components and rules of interactions are *non-linear* .... **The computer plays a critical role in this research**, because it becomes impossible to say much directly about the dynamics of non-linear systems with a large number of degrees of freedom using **classical mathematical analytical methods**.

ibid, p.2; bold emphasis added.

Note, however, that the discourse is about *potentials* and *tendencies* and, therefore, in an economic context, but not only in it, there could be scope for *design* or *policies*. Moreover, the 'avoidance of stable, self-reproducing states' is an indictment against mechanical growth theories, of a macroeconomic sort, with their uninteresting, stable, attractors.

This is exactly where the notion of *computational irreudicibility*, interpreted in economic contexts as *computation universality*, especially within what I have come to call *Computable Economics*[7], plays an important role, both at the level of individual behaviour and institutional design - or, both microeconomically and macroeconomically. In the former case, i.e., microeconomically, at the level of individual behaviour, the notion allows one to show that the much maligned Simonian concept of Bounded Rationality encapsulates, naturally and generally, the normal practice of rational behaviour; in the latter case, i.e., macroeconomically, it allows one to derive an impossibility theorem on policy.

One final cautionary note has to be added, lest the unwary practitioner of indiscrimnate reliance on 'pattern recognition' of computer graphics is lulled into thinking that the pixels on the screen are independent of the mathematics of the computer – i.e., recursion theory. It is necessary to remember the following classical result from recursion theory:

Letφ and ζ be a partial and a total function, respectively. Then:

- φ is partial recursive iff its graph is *recursively enumerable*.
- ζ is recursive iff its graph is *recursive*.

This kind of result is alien to the practitioners of varieties of agent-based modelling, who seem to rely on blind computer graphics for significant inductive propositions. Stephen Wolfram never made such a mistake – nor did, naturally, Turing!

[7] A brief, but rigorously intuitive – in the senses in which the *Chruch-Turing Thesis* can be referred to in this way – characterisation of what I mean by *Computable Economics* can be gleaned from (Kao, et.al., 2012). A more detailed, but already dated, characterisation is in Velupillai (2000).

2 Computational *Irreducibility* as Computation *Universality*, implied by the Principle of Computational *Equivalence*.

> "And what the *Principle of Computational Equivalence* implies is that in fact almost any system whose behaviour is not obviously simple will tend to exhibit *computational irreudicibility.*"
>
> **NKS**, p. 745; italics added.

I have, within the framework of *Computable Economics*, attempted to formalise the notion of *computation universaity*, as against *computationally irreducible*, in terms of the computing *trajectories* of finite automatons and Turing Machines, respectively, with the formalisation of the notion of *trajectories* in terms of formal *dynamical systems.* Moreover, my definition of the dynamics of a *complex* economic system as one capable of computation universality was, I realised with the benefit of hindsight, exactly similar to a 'system whose behaviour is *not obviously simple*' and, therefore, one which 'will tend to exhibit computational irreducibility'. What I have claimed, explicitly, is an equivalence between the notion of *complex* and *not obviously simple.* I go further and assert that it is not possible to prove – by means of any notion of proof – a formal equivalence between the notion of *complex* and the phrase *not obviously simple* – except, for example, by means of invoking something like computationally *reducible* and enunciating a thesis that *simple* is equivalent to *reducible*[8]. It was, then, straightforward to identify the behaviour of a finite automaton with one that is computationally reducible, for example one that is not subject to the *Halting problem for Turing Machines* or, more pertinently, one that is only capable of *trivial* dynamics, with the notion of 'trivial' given a formalism via *Rice's Theorem.* In this way *simplicity*, *reducibility* and *triviality* can be coupled to the computing dynamics of *finite automata* and *complexity*, i*rreducibility* and *non-triviality* to those trajectories that are routine for a *Turing Machine.*

Finally, Wolfram's characteristically fertile assumption, one which I have come to call **Wolfram's Thesis (NKS**, p. 715)

Claim 1 *All processes can be viewed as computations*

I claim that this is a kind of 'dual' to the following variant of the Church-Turing Thesis (cf. Beeson, p. 34):

Claim 2 *Every rule is a recursive rule*

With this conceptual background at hand, I can illustrate the derivation of two fundamental results in microeconomics and macroeconomics – one on the

[8] Or the notion of simplicity to be formally identified with *reducibility in a computational sense.* I am convinced that this corresponds exactly to the way Kemeny used the notion of *simplicity in induction* in his classic contribution which lies at the basis of *algorithmic complexity* (Kemeny, 1953; Li & Vitanyi, 1997).

notion of rational behaviour and the other on the feasibility of effective policy in a complex (dynamic) economy.

Definition 3 ***Invariant set***

A set (usually compact) $S \sqsubset U$ is ***invariant*** *under the flow $\varphi(.,.)$ whenever $\forall t \in \mathbb{R}$, $\varphi(.,.) \sqsubset S$.*

Definition 4 ***Attracting set***

A closed invariant set $A \sqsubset U$ is referred to as the ***attracting set*** *of the* ***flow*** *$\varphi(t,x)$ if $\exists$ some neighbourhood V of A, s.t $\forall x \in V$ & $\forall t \geq 0$, $\varphi(t,x) \in V$ and:*

$$\varphi(t,x) \to A \text{ as } t \to \infty \tag{1}$$

Remark 5 *It is important to remember that in dynamical systems theory contexts the attracting sets are considered the* ***observable*** *states of the dynamical system and its flow.*

Definition 6 *The basin of attraction of the attracting set A of a flow, denoted, say, by Θ_A, is defined to be the following set:*

$$\Theta_A = \cup_{t \leq 0} \varphi_t(V) \tag{2}$$

where: $\varphi_t(.)$ denotes the flow $\varphi(.,.), \forall t$.

Remark 7 *Intuitively, the basin of attraction of a flow is the* ***set of initial conditions*** *that eventually leads to its attracting set - i.e., to its limit set (limit points, limit cycles, strange attractors, etc).*

Definition 8 ***Dynamical Systems capable of Computation Universality:***

A dynamical system capable of computation universality is one whose defining initial conditions can be used to program and simulate the actions of any arbitrary Turing Machine, in particular that of a Universal Turing Machine.

Proposition 9 *Dynamical systems characterizable in terms of limit points, limit cycles or 'chaotic' attractors, called 'elementary attractors', are not capable of universal computation.*

Theorem 10 *There is no effective procedure to decide whether a given observable trajectory is in the basin of attraction of a dynamical system capable of computation universality*

Proof. The first step in the proof is to show that the basin of attraction of a dynamical system capable of universal computation is recursively enumerable but not recursive. The second step, then, is to apply Rice's theorem to the problem of membership decidability in such a set. First of all, note that the basin of attraction of a dynamical system capable of universal computation is recursively enumerable. This is so since trajectories belonging to such a dynamical system can be effectively listed simply by trying out, systematically, sets of appropriate initial conditions. On the other hand, such a basin of attraction is not recursive. For, suppose a basin of attraction of a dynamical system capable of universal computation is recursive. Then, given arbitrary initial conditions, the Turing Machine corresponding to the dynamical system capable of universal computation would be able to answer whether (or not) it will halt at the particular configuration characterising the relevant observed trajectory. This contradicts the unsolvability of the Halting problem for Turing Machines. Therefore, by Rice's *theorem*, there is no effective procedure to decided whether any given arbitrary observed trajectory is in the basin of attraction of such recursively enumerable but not recursive basin of attraction. ■

Given this result, it is clear that *an effective theory of policy is impossible in a complex economy.* Obviously, if it is effectively undecidable to determine whether an observable trajectory lies in the basin of attraction of a dynamical system capable of computation universality, it is also impossible to devise a policy - i.e., a recursive rule - as a function of the defining coordinates of such an observed or observable trajectory. Just for the record I shall state it as a formal proposition:

Proposition 11 *An effective theory of policy is impossible for a complex economy*

Remark 12 *The 'impossibility' must be understood in the context of effectivity and that it does not mean specific policies cannot be devised for individual complex economies. This is similar to the fact that non-existence of general purpose algorithms for solving arbitrary Diophantine equations does not mean specific algorithms cannot and have not been found for special, particular, such equations.*

What if the realized trajectory lies outside the basin of attraction of a dynamical system capable of computation universality and the objective of policy is to drive the system to such a basin of attraction? This means the policy maker is trying to design a dynamical system capable of computational universality with initial conditions pertaining to one that does not have that capability. Or, equivalently, an attempt is being made, by the policy maker, to devise a method by which to make a Finite Automaton construct a Turing Machine, an impossibility. In other words, an attempt is being made endogenously to construct a 'complex economy' from a 'non-complex economy'. Much of this effort is, perhaps, what is called 'development economics' or 'transition economics'. Essentially, my claim is that it is recursively impossible to construct a system capable of computation universality using only the defining characteristics of a Finite Automaton. To put it more picturesquely, a *non-algorithmic step* must be taken to go from systems *incapable* of self-organisation to ones that are capable of it. This interpretation is entirely consistent with the original definition, explicitly stated, of an '*emergent property*' or an '*emergent phenomenon*', by George Henry Lewes. This is why 'development' and 'transition' are difficult issues to theorise about, especially for policy purposes.

Next, consider a (rational) *problem solving* entity to be an *information processing system* (Newell & Simon, 1972). The strategy for my formalization exercise can be summarized in the following sequence of steps:

- Extract the procedural content of orthodox rational choices (in theory).
- Formalize such a procedural content as a process of computation.
- Given the formalized procedural content as a process of computation, to be able to discuss its computational complexity.
- Show the equivalence between a process of computation and a suitable dynamical system.
- To, then, show the possibility of non-maximum rational choice.
- Then, to show that such behaviour is that which is manifested by a boundedly rational, satisficing, agent.

The following results encapsulates, formally, the content of the first three steps of the above six-step scheme:

Theorem 13 *The process of rational choice by an economic agent is formally equivalent to the computing activity of a suitably programmed (Universal) Turing machine.*

Proof. By construction. See §3.2, pp. 29-36, Velupillai (2000). ■

Remark 14 *The important caveat is 'process' of rational choice, which Simon – more than anyone else – tirelessly emphasized by characterizing the difference between 'procedural' and 'substantive' rationality; the latter being the defining basis for Olympian rationality (cf. Simon, 1983, chapter 1), the former that of the computationally underpinned problem solver facing decision problems. Any decision – rational or not – has a time dimension and, hence, a content in terms of some process. In the Olympian model the 'process' aspect is submerged and dominated by the static optimization operator, By transforming the agent into a problem solver, constrained by computational formalisms to determine a decision problem, Simon was able to extract the procedural content in any rational choice. The above result is a summary of such an approach.*

Theorem 15 *Only dynamical systems capable of computation universality are consistent with rationality in the sense that economists use that term in the Olympian Model.*

Proof. See pp. 49-50, Velupillai (2000). ■

Remark 16 *This result, and its proof, depend on theorem 13 and, therefore, its background basis, as explained in the Remark following it, given above. In this way, following the Simon's vision and the definition of rationality is divorced from optimization and coupled to the decision problems of an information processing problem solver, emphasizing the procedural acts of choice.*

Theorem 17 *Non-Maximum Rational Choice*

*No trajectory of a dynamical system capable of universal computation can, in any '**useful sense**' (read: '**cannot obviously**') be related to optimization in the Olympian model of rationality.*
Proof. See Velupillai (2007). ■

Remark 18 *The claim here is, then, that optimization in the Olympican model of rationality is computationally reducible.*

Theorem 19 *Boundedly rational choice by an information processing agent within the framework of a decision problem is capable of computation universaltiy.*

Proof. An immediate consequence of the definitions and theorems of this section. ■

Remark 20 *From this result, in particular, it is clear that the Boundedly Rational Agent, satisficing in the context of a decision problem, encapsualtes the only notion of rationality that can 'in any useful sense' be defined procedurally.*

3 A Computable Economist's *Paen* to the NKS Vision[9]

> "Mathematics is not a finished object based on some axioms. It *evolves* genetically. This has not yet quite come to conscious realization. ...
>
> *Mathematics will change. Instead of precise theorems*, of which there are now millions, *we will have, fifty years from now, general theories and vague guidelines*, and the individual proofs will be worked out by graduate students or by computers.
>
> *Mathematicians fool themselves when they think that the purpose of mathematics is to prove theorems, without regard to the broader impact of mathematical results.* Isn't it strange.
>
> *In the next fifty years there will be, if not axioms, at least agreements among mathematicians about assumptions of new freedoms of constructions, of thoughts.* Given an undecidable proposition, there will be a preference as to whether one should assume it to be true or false. Iterated this becomes: some statements may be undecidably undecidable. This has great philosophical interest
>
> Ulam, in Cooper (1989), pp.310-312; italics added.

[9] The *paen* of this section is a slightly modified version of an initial attempt to pay *homage* to **NKS**, in Velupillai (2005).

Ulam was always a prescient mathematician. In an almost uncanny confirmation of his audacious prediction, Stephen Wolfram's NKS seems to have set out an implementable version of the vision of the kind of *mathematics* – but not quite *science* – that Ulam may have had in mind. It is particularly appropriate that Wolfram achieved this implementable vision of Ulam's vision of *A New Kind of Mathematics* utilising Cellular Automata as his computational paradigm – especially since it was Ulam, together with von Neumann, who pioneered the use of this medium for the kind of questions that are fundamental in the *new sciences of complexity*. Indeed, Wolfram's explicit philosophical and epistemological stances on the nature and evolution of mathematics buttresses my own vision of the mathematical foundations of Computable Economics:

> "[L]ike most other fields of human enquiry mathematics has tended to define itself to be concerned with just those questions that its methods can successfully address. And since the main methods traditionally used in mathematics have revolved around *doing proofs*, questions that involve *undecidability* and *unprovability* have inevitably been avoided. ...
>
> The main point .... is that in both the systems it studies and the questions it asks mathematics is much more a product of its history than is realised."
>
> **NKS,** p. 792; italics added

It is precisely these questions that have *not* been avoided in *Computable Economics* and precisely because, inadvertently, those of us who were busy developing this alternative vision of doing economics in the mathematical mode underpinned our methodologies and epistemologies in what, with hindsight, could be identified with the Principle of Computational Equivalence (**PCE**), the notion of computational irreducibility and Wolfram's Thesis – albeit in other, but equivalent, formalisms.

If economics is formalised using traditional mathematics, it will be crippled by the poverty of the history that determines that tradition – but it may also be enhanced by the richness of that tradition. It is just that the richness seems an entirely internal history, completely unrelated to the ontology of economics. This, I think, is the point made by Ulam and Wolfram in their indictments of the traditional philosophy and epistemology of mathematics and science, respectively. Economics in the mathematical mode, a step child and a handmaiden to both of these endeavours, in applying the concepts and methods emerging (sic!) from their traditional evolutionary history, would therefore by crippled to the same extent as these noble products of the human mind.

Essentially, computational irreducibility of a computational process, by **PCE** and **Wolfram's Thesis**, implies also *unpredictability*[10]. Wolfram is then able to show that traditional science is computationally *reducible* and, hence, can succeed in local predictability. This, in turn, is because the mathematical laws that

[10] To the triptych of *undecidability*, *unprovability* and *unpredictability* must be added *unsolvability* and *uncomputability* as the five cardinal unifying conceptual bases that underpin *Computable Economics* (cf. Kao, et. al., 2012).

encapsulate traditional science are simple enough to be analytically tractable and, hence, are computationally reducible.

In other words, he extracts the implicit processes intrinsic to, and implied by, any mathematical law or formalism in traditional science, and uses the **PCE**, and its phenomenological consequence – computational irreducibility – to evaluate their effective and predictable content to show the simplistic, unrealistic and undesirable nature of mathematical formalism in the traditional sciences.

I believe **NKS** succeeded admirably in showing that the appearance of the successes of traditional science, using the criteria of simplicity in the laws that encapsulate the phenomena to be explained and the predictions that are extracted from them, are the results of a subterfuge: that of concentrating on using analytically solvable mathematical formalisms to encapsulate natural laws and thereby ignoring the *processes* that have to be used to evaluate the solution to any such formalism. In this way, the methodology of traditional science circumvented the implications of computational irreducibility and, hence, ignored **PCE**.

But he may have forgotten that Brouwer could have also been, together with Ulam, a compatriot – with his *Choice Sequences*, which enriching **PCE** and *computational irreducibility* in ways that would have enhanced the positive aspects of his negative results.

All this could have been said, *pari passu*, of the achievements - or, more accurately put, the *non-achievements* – of traditional mathematical and analytical economics.

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