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NONCOMPUTABILITY, UNPREDICTABILITY, UNDECIDABILITY & UNSOVABILITY IN ECONOMIC & FINANCE THEORIES♣

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♣Professor Daniel S. Graça kindly sent us the `internet' version of his paper, *Noncomputability, Unpredictability, and Financial Markets*, just published in **Complexity**. In a footnote to the Internet version he handsomely acknowledges that his assertion in the published version of this interesting paper that `no noncomputable problem with economical or financial inspiration has been presented before' is `incorrect'. He, then, generously mentions the work of our group on *Computable Economics* (although the inclusion of J.Holm in this list is slightly anomalous) as evidence towards a correction of his `assertion'. This note is simply a minor note of clarification, amplifying the `assertion' and providing more specific examples - directly relevant to the content of Professor Garça's stimulating contribution. Quite apart from the particular inspiration of the general issue discussed in Professor Garça`s paper, and in ours', we have benefited greatly from his interesting work on computability in dynamical systems (cf. for example, Collins & Garça, 2008 and Ragupathy & Velupillai, 2012)

Abstract

We outline, briefly, the role that issues of the nexus between noncomputability and unpredictability, on the one hand, and between undecidability and unsolvability, on the other, have played in *Computable Economics*. The mathematical underpinnings of Computable Economics are provided by (classical) recursion theory, varieties of computable and constructive analysis and aspects of combinatorial optimization. The inspiration for this outline was provided by Professor Garça's thought-provoking recent article in this Journal.

1 Noncomputability, Unpredictability, Undecidability and Unsolvability in Economics & Finance

Professor Garça, in his thought-provoking article in this Journal, has raised the important question of the nexus between *noncomputability* and *unpredictability* in economics and finance theoretical frameworks. This nexus, together with that between *undecidability* and *unsolvability*, has been at the core of classical behavioural economics ([12]), arithmetical and orthodox game theory, choice theory, learning and dynamics in micro and macro economics and in many other core areas of economics and finance theory, for almost exactly half-a-century. These are issues that have been at the core of research in what we have come to call *Computable Economics*[1] from its very inception, serendipitously, at the hands of one of the foremost pioneers of computability theory, Alan Turing ([25][2]). Turing's classic formalisation of 'solvable and unsolvable problems' – his last published paper – was one of the crucial starting points[3] for the emergence of classical behavioural economics, almost single-handedly forged by Herbert Simon[4], within the framework of *Human Problem Solving* ([15]), underpinned by *boundedly rational agents* seeking *satisfactory* solutions to economic decision problems[5]. Indeed, the doyen of mathematical economics, Kenneth Arrow, emphasised the nexus identified by Professor Graça – between *noncomputability* and *unpredictability* – at about the half way point between Turing's classic of 1954 and **CE** in a characteristically prescient conjecture ([2][6], p. S398; italics added):

"The next step in [economic] analysis, I would conjecture, is a more

[1] *Computable Economics* as a new name, characterising the formalisation of economic and finance theoretic entities using the formalism of recursion theory and constructive analysis, was coined by Velupillai in 1983. At that time he used it interchangeably with Turing's Economics, but settled on Computable Economics in 1987.

[2] Reprinted, together with one of Kleene's classics, as the lead article in [31]. We shall refer to this collection as **CE** in the sequel.

[3] The others being Polya's approach to problem solving, as first formulated in the elegant How to Solve It? ([17]) and Peirce's fertile concept of retroduction (as distinct from the tiresome dichotomy between induction and deduction), going back to Bolzano's idea of abduction, which itself can be traced - as all such things can - to Aristotle.

[4] Herbert Simon is the only person to win both the Nobel (Memorial) Prize in economics (1977) and the Turing Prize (1975), awarded by the ACM.

[5] Decision problems in the strict sense of metamathematics and, hence, leading to the modern framework of computational complexity theory, to the empirical development of which Herbert Simon contributed significantly. It must be remembered that Simon was not only a Professor of Economics, but also of Computer Science and (Cognitive) Psychology.

[6] In more senses than one this was the year that Computable Economics 'came of age', with the comprehensive research program set out by Alain Lewis ([13]). For reasons that we have been unable to comprehend, this fertile research program was inexplicably abandoned a few years later by Alain Lewis. It may be useful to add that Velupillai was unaware of the research program Lewis had formulated at the time he coined the name Computable Economics. Moreover, Velupillai was informed by Paul Samuelson, during a personal telephone conversation in January 1987, that Kenneth Arrow had been a student of Emil Post and that Alain Lewis considered himself a pupil of Kenneth Arrow.

consistent assumption of *computability* in the formulation of economic hypotheses. This is likely to have its own difficulties because, of course, *not everything is computable*, and there will be in this sense *an inherently unpredictable element* in rational behavior."

The complete analytical framework for this particular nexus, within *Computable Economics*, was elegantly summarised by [10].

As for the modelling of economic dynamics, both at the macro and micro economic level, using one or another variety of computable analysis, in particular invoking what he called the Pour El - Richards theorem ([18]), but with explicit references to both [1] and [14], Velupillai, in his *Arne Ryde Lectures* of 1994[7], made the relevance of recursively inseparable sets in generating uncomputable solutions in perfectly orthodox settings of economic problems – but pointing out the need for refining the definition of the relevant domain to be sets of natural (or rational or, for reasons of analytical necessity, algebraic numbers).

Finally, there is the relevance of *Hilbert's Tenth Problem* and the *Halting Problem for Turing Machines* in typical economic settings. The former was comprehensively surveyed in [26]; (chapter 19 of **CE**), with an especial place for the pioneering work of [19]; (see **CE**, chapter 14), for the whole tradition of undecidability and unsolvability in arithmetical games properly assigned. The latter – i.e., the *Halting Problem for Turing Machines* – was elegantly invoked in a computable macroeconomic model of growth, framed as a *Busy Beaver Game*, by [33]; (chapter 28 of **CE**)[8].

In the case of *computable finance theory*, the current classic is, of course, the comprehensive textbook by [22], but one of the undisputed pioneers of finance theory and its empirical analysis, Maury Osborne [16], was fully aware of the nexus between noncomputability and undecidability emerging from perfectly ordinary and orthodox financial data[9]. Within our own work, as Computable Economists, Shu-Heng Chen's doctoral dissertation [8], written under Velupillai's supervision, at the *Center for Computable Economics* at UCLA (now defunct), explored actual data generated in the functioning of the Taiwanese stock market using the framework of Kolmogorov (or algorithmic) complexity to investigate uncomputability and its implications for the efficient market hypothesis.

Quite apart from any variety of computable analysis[10], there is the fertile area of constructive analysis where the noncomputability-unpredictability nexus emerges via a consideration of choice sequences in intuitionistic mathematics [24], an area of research in Computable Economics that is only now

[7]Published much later as [28], due to an unforgivable error by Oxford University Press, to whom the manuscript was submitted in 1998! Meanwhile, the results appeared in [27].

[8]Contrary to what is asserted by Garça as an explanatory comment in stating his 'Proposition 6' (ibid, p. 4), Zambelli was able to use the *Busy Beaver Function* to generate noncomputable numbers in exact analogy with [20], thereby avoiding the use of any nonconstructive diagonal argument.

[9]The 'stock market charts' to which Professor Garça correctly refers.

[10]Professor Garça's article works within the particular framework of *Computable Analysis* elegantly developed by [32], itself based on what we have, in other writings, referred to as the '*Polish Tradition*' of *recursive analysis* [3]. [1], for example, is in what we refer to as the *Russian* – or Markov – *tradition* of *Computable Analysis*.

being developed. However, the place of constructive analysis in Bishop's sense [4], and its relevance for core areas of microeconomic choice theory, was elegantly and almost comprehensively analysed by Bishop's co-author, Douglas Bridges at a very early stage of the evolving research program of computable economics ([6, 7]; Chapters 8 & 9, in **CE**). An insight into a noncomputable result and a constructive proof, the fertile interaction between classical recursion theory and basic Bishop-style constructive analysis, is provided by the demonstration that the foundational result of *Computable General Equilibrium Theory*, the *Uzawa Equivalence Theorem*, implies the decidability of the *Halting Problem for Turing Machines* ([29]; chapter 24 of **CE**) and the *Lesser Limited Principle of Omniscience* (**LLPO**; [23]; chapter 25 of **CE**).

2 Brief Clarifying Notes on the Economics of Graça's Analysis

Professor Graça, in our opinion, raises important methodological issues on the role of mathematical formalism in economics and finance theories. To make his serious methodological points concrete he tries to frame the issue of the nexus between noncomputability and unpredictability in terms of a simple economic problem: that of discounting a stream of returns using a standard formula. Lest the unwary economics reader of his important methodological paper loses sight of the rich thicket that are the woods, for the fragile trees, we would like to clarify some of the possible misunderstanding due to infelicities in the economic arguments.

Right at the outset it must be pointed out, as clearly indicated in the rich vein of results reported in the previous section, Professor Garça's important methodological insights on the nexus between noncomputability and unpredictability in economics and finance, in no way depends on the particular economic and financial examples in which he clothes his discussion.

The main infelicities, however, are the following: the reference to a 'fair value' in the discounted cash flow model (DCFM); the connection of this model with the Capital Asset Pricing Model (**CAPM**), is *less than tenuous*; the key formal, analytic, assumption that $e : \mathbb{N} \to \mathbb{R}$ and $u : \mathbb{N} \to \mathbb{R}$ are never made in economics – indeed, neither are the members of the range of any such mapping, in economic contexts, are restricted to any kind of computable reals and, hence, almost all discussions of approximating any kind of equilibrium value in economics or finance theories are meaningless; the example of the German hyperinflation of 1923 as showing the difficulty of prediction, referring to a chart in Bresciani-Turroni's classic ([5], p. 39), is slightly misplaced.

If we take just the last example, Garça – who is not an economist and, therefore, should not be 'taken to task' for any of the economic or finance theoretic infelicities – misses the important point, which any reasonable macroeconomist would have inferred, that the astronomical inflationary levels were themselves entirely predictable; the difficulty was predicting the exact point at which 'the mark was clearly detached from the group of other principal European currencies'. As Bresciani-Turroni (*ibid*, p. 38). perceptively observed:

> "Throughout this period [1914 – second half of 1919] the movement of the mark exchange was analogous to that of the other principal European exchanges, save for the greater amplitude of fluctuation."

The 'detachment' of this 'movement of the mark exchange', from about the second half of 1919 is, then, the subject of monetary macroeconomic theorising and the onset of the kind of hyperinflation that was observed, and its similarity with more modern episodes in Africa and Latin America, was entirely predictable once the point of detachment was identified (even if not its reasons, except with hindsight).

No discounted cash flow formula has anything to do with a 'fair value', in any formally definable sense of 'fairness' in economics (or finance).

Professor Garça's entirely correct intuition to assume that the domain of the earnings (e) and discount (u) is $\mathbb{N}$ is not part of the usual repertoire of assumptions of the economic or financial modeller – despite facing only members of $\mathbb{N}$ (or, at best, $\mathbb{Q}$) represented on 'stock market charts'. This was an observation made by Maury Osborne at the dawn of mathematical modelling of financial market data, but the profession has chosen to ignore this important fact and return to what may be called the '*Bachelier tradition*' ([11]), of exploring and exploiting, indeed flogging to pointless death, the nexus between *non-algorithmic randomness* and unpredictability.

3 Whither Computable Economics?

Professor Garça has raised fundamental doubts on the absolutely important methodological relevance of tying, almost indissolubly, a *Gordian knot* between unpredictability and non-algorithmic randomness. One of the cardinal methodological precepts of *Computable Economics* has been to cut this – and many other orthodox – *Gordian knot(s)*, in particular, as in Professor Garça's insightful attempt at exploiting the nexus between noncomputability and unpredictability, by formalisms of fundamental economic and financial entities and units in terms of recursion theory, computable analysis and constructive mathematics.

Our basic insight leads us to believe that Herbert Simon's original research program, which we now refer to as Classical Behavioural Economics, when supplemented by the methods of computable analysis and constructive mathematics provides a fuller justification of economic and financial decision making – and their outcomes as institutional evolution and market data – as decision problems in the strict sense in which this is defined in metamathematics and combinatorial mathematics. Essentially, the methodological and epistemological implications of such a stance in the mathematical modelling of economics and finance theories is that there will be, as Arrow (*op.cit*) perceptively observed more than a quarter of a century ago, an '*inherently unpredictable element in rational behaviour*'.

It must be emphasised that *none* of this has anything to do with '*Butterfly effects*' and other fashionable *pseudo-unpredictabilities* due to *sensitive dependence on initial conditions* (SDIC), horse shoes, and such deterministic nonlinear dynamical formalisms - but they have everything to do with the *Incompleteness* intrinsic to *Goodstein's Algorithm* and its related results invoking *Ramsey Theory*. But this is quite another story - of *ineffective, incomplete and undecidable policy games* in rich economic dynamics ([30]).

In this sense, then, Professor Garça's contribution should be welcomed by all computable economists. It is our hope that it will also be welcomed by those who still choose to work with stone age tools and invoke related Paleolithic epistemologies.

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