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ZERMELO, VON NEUMANN, EUWE AND *ZERMELO'S THEOREM ON CHESS*

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Zermelo, von Neumann, Euwe and *Zermelo's Theorem on Chess*♣

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♣ Aumann's '*Underground Classic*', **Lectures on Game Theory** (Aumann, 1989, p.1), begins chapter 1, itself titled *Zermelo's theorem*, with a proof of **Zermelo's Theorem on Chess**. Although I have not sought his approval, I would very much like to dedicate this brief essay to Bob Aumann on the occasion of his 85th birthday, with fond memories of having been his office neighbor, at C.O.R.E during the academic year 1977-78. My remembrances go back to discussions on Veronese's priority on non-standard analysis, as I was reading the first volume of Hobson's *Theory of functions of a Real Variable* and Aumann's translation of Hausdorff's classic on **Set Theory**. The translator(s) were named as 'John. R. Aumann, ET AL,.', on the title page and I was puzzled at the ordering of the initials. Bob Aumann, with disarming modesty, almost sheepishly, admitted he was 'John. R Aumann', the translator of Hausdorff1

Abstract

Zermelo is important in mathematical economics and game theory not only for the *well-ordering theorem*, in the guise of the *Axiom of Choice* or *Zorn's Lemma*; and also for the standard framework of *ZFC* – i.e., Zermelo-Fraenkel set theory with the axiom of choice. He is, increasingly, also known for his celebrated 1912/1913 theorem on chess. However, it is claimed that von Neumann, the founding father of game theory, did not ever refer to Zermelo, in this particular context of chess-like games, in the generally known literature in the fields. Moreover, there are claims that Zermelo provided an algorithm for implementing his theorem by means of his proof. This note attempts to clarify both these claims.

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§ 1. A Preamble – Some Historical Notes

> "J. von Neumann was aware of the importance of the minimax principle [von Neumann, 1928]; it is, however, difficult to understand the absence of a quotation of Zermelo's lecture in [von Neumann's] publications."
> Steinhaus, 1965, p. 460

Steinhaus is referring to Zermelo (1913)[1], which is now acknowledged as *the* pioneering work in what has come to be called Game Theory. In a comprehensive technical and largely complete historical introduction to Zermelo (ibid), Larson[2] (2009, p. 265; bold italics added), reinforces the standard view on references to this classic:

> "Aside from the work of König and Kalmar, the ***earliest*** citation of Zermelo's *1913* that we have been able to find is Kuhn's *1953*.

Yet, in van Dalen's admirably complete biography of Brouwer (van Dalen, 2005, p. 636; italics added) it is stated quite categorically that:

> "In 1929 there was another publication in the intuitionistic tradition*: an intuitionistic analysis of the game of chess* by Max Euwe [(1929)]. It was a paper in which the game was viewed as a *spread* (i.e. a tree with the various positions as nodes)[3]. Euwe carried out *precise constructive estimates* of various classes of games, and considered the influence of the rules for draws. When he wrote his paper he was not aware of the earlier literature of Zermelo and Dénès König. *Von Neumann called his attention to these papers*, and in a letter to Brouwer Von Neumann sketched a classical approach to the mathematics of chess, pointing out that it would *easily be constructivized[4]*."

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[1] This classic by Zermelo is sometimes cited as Zermelo (1912), which is not entirely incorrect. It was a lecture given by Zermelo in 1912, but the first publication containing it was in Zermelo (1913).

[2] Who notes (ibid, p. 263) that 'König, already in 1926, credits' von Neumann for the 'application of [the *König Lemma*]' to clarifying some of the 'looseness' in Zermelo (1913).

[3] This is the Brouwerian constructive mathematical equivalent of the standard formulation of extensive form games.

[4] von Neumann *never constructivized* the 'classical approach to the mathematics of chess – nor is it true that 'it would easily be constructivized' (see the next section). In fact, to the best of this author's knowledge, von Neumann never 'constructivized' any proof of any of his game theoretic or growth theoretic theorems (despite claims to the contrary in the published literature, for example, Giocoli, 2003). The excellent survey of von Neumann's work in game theory and the mathematical theory of growth by Kjeldsen (2001), contains a good exposition of the evolution of the mathematical basis of the theorems von Neumann 'proved', in these two fields. I have some reservations about at least two of the claims in this fine contribution by Kjeldsen (that on mathematical programming and the opinion on Mirowski, but these are very minor disagreements). I am slightly bemused that Kjeldsen translates *Gesllsschaftsspiele* (most likely

von Neumann's strange reticence to refer to Zermelo (1913), in any of his well-known game theoretic papers (von Neumann, 1928a, 1928b[5], 1953b) or in von-Neumann-Morgenstern (1944 [1947,1953]) remains, to this day, unresolved[6]. Unlike his interchange with Frechet on Borel (von Neumann, 1953a), mostly precipitated by others, it is now too late to engage von Neumann in a debate on Zermelo's priority on a *set theoretic analys*is of Chess-like (parlour) games.

The title of Zermelo's classic of 1913 is: *Ü ber eine Anwendung der **Mengenlehre** auf die Theorie des Schachspiels*. Now contrast this with the title of Euwe's explicitly acknowledged, Brouwerian intuitionistic analysis of the playing of the game of chess: ***Mengentheoretische** Betrachtungen über das Schachspiel*[7]. In the former case, the reference is to classical**, ZFC**[8], set theory; in the latter case it is to **Intuitionistic** set theory. Without an understanding of the precisely different mathematical foundations – the *metamathematics* – of the two notions, the reader of the latter would not have a clue as to the nature of the min-max derived, with precisely exact numerical approximation, by Euwe.

At the minimum, even a sympathetic reader of Euwe's classic, will need to acquaint herself with the rudiments of Intuitionistic mathematics and its metamathematics. This means, at least to read this paper by Euwe with some mathematical competence, some minimal acquaintence with the precise intuitionistic notions of the 'triptych' of (Brouwerian constructive notion of) *sets*, *spreads* and *proof*.

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following von Neumann, 1959) as *Games of Strategy*, rather than as *Parlour Games* (cf. Kjeldsen, ibid, p. 42).

[5] It must be remembered that this paper was presented in Göttingen on *7th December, 1926* (cf. von Neumann, 1928, footnote 1, p. 265; the English translation of this classic is von Neumann, 1959). This *date* alone casts doubts on Mirowski's claims (cf. Kjeldsen, op.cit, p.42). Hilbert's sharpening of his 'program' was precipitated not only by Brouwer's critique and alternative, intuitionistic constructive, program; it was also due to the stance taken by Weyl (1921). Thus, the Hilbert Program could be traced to an origin in the early 1920s (see, Hilbert, 1921), but it crystallized and was codified only in the late 1920s.

[6] It is even more strange that the only two references to Zermelo, in von Neumann-Morgentern are to that great mathematician's contribution to classical set theory, via the well-ordering principle (von Neumann-Morgenstern, ibid, pages 269 & 595.

[7] In both cases the boldface emphasis has been added.

[8] The '**Z**' in **ZFC** stands, of course, for **Zermelo**! **F** and **C** stand for **Fraenkel** and the *Axiom of **Choice***, respectively.

A clear discussion of the first of these concepts can be found in Veldman (1990), but, of course, the classical exposition is that of Brouwer (1919); the classic exposition of the intuitionistic notion of *Spreads* is in Brouwer (1981, p. 15, ff); and, of course, there is no better place to acquaint oneself with the *intuitionistic notion of Proof* than in Brouwer (1908).

Euwe's paper, and its *constructive* results, including a *constructive mini-max theorem*, extracted from his doctoral dissertation under Brouwer, was communicated to von Neumann in 1929, which elicited the latter's response, mentioned above.

Euwe (1929)[9] states, in an addendum [**§ 5 ***] to his own pioneering constructive approach to Chess:

> "After completing the preceding observations, I was informed by Dr. NEUMANN about two other works dealing with the same topic [those of Zermelo, *op. cit.*, and König, 1927].

I suppose the best reaction in this kind of case is to recall Stigler's rue remark (1966, p. 77):

> "If we should ever encounter a case where a theory is named for the correct man, it will be noted."

In the next section some of the constructive and algorithmic claims on the proof(s) of *Zermelo's Theorem on Chess* are presented. The third, concluding, section is an attempt at summarizing the 'story'.

§ 2. Zermelo's Theorem on Chess[10]

> "In the finite case, although classically we can assert that machines *exist* which compute characteristic functions, and in individual cases we *may* be able to find such machines, there is *no (partial) recursive method* of generating them from the Gödel numbers of the 'increasing order' of machines."
> Harrop, 1961, p. 140; italics added.

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[9] The quotation is from the translation forthcoming in **New Mathematics and Natural Computation** (2015). Max Euwe, a few years later, became the World Chess champion, defeating Alekhine.

[10] Aumann's '*Underground Classic*', **Lectures on Game Theory** (Aumann, 1989, p.1), begins chapter 1, itself titled *Zermelo's theorem*, with a proof of **Zermelo's Theorem on Chess**.

Harrop's celebrated theorem (*ibid*, p. 136) is on the possibility of constructing algorithms to *decide* properties of *finite sets*. All of the proofs of *Zermelo's Theorem on Chess* in the standard game theoretic literature – at least so far as I know them[11] – invoke one or another form of mathematical induction[12], and apply it in its supposedly practical form of 'backward induction'. But the 'warning' by Megiddo and Wigderson (1986, p. 260; italics added) is, generally, not heeded in the wide and sweeping claims of implementation, even in principle:

> "It should be emphasized that *most* of the results [using *induction in proofs*] are *nonconstructive* and we do *not* believe that the approach taken in this paper will lead to a practical resolution [of games played by machines]"

Thus, in a widely used textbook on Game Theory, we can read (Osborne & Rubinstein, p. 6; italics added) that:

> "[C]hess is a *trivial game* for 'rational' players[13]: an *algorithm exists* that can be used to 'solve' the game. This *algorithm* defines a pair of strategies, one for each player, that leads to an 'equilibrium' outcome .. . The *existence* of such strategies suggests that chess is uninteresting because it has only one possible outcome. … Its equilibrium outcome is yet to be *calculated*; currently it is impossible to do so using the algorithm."

Some comments on the statements by Osborne and Rubinstein are in order, at least for the sake of clarifications. Firstly, *existence* of an *algorithm*, by itself, does not make an implementation

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[11] Euwe (*op.cit*) is, of course, the notable exception.

[12] Aumann (op.cit, p. 2)) and Hart (1992, pp. 30-1). Aumann's proof is crisp and concise; Hart's proof is more explanatory – but words like 'determined' (admittedly within quotes), construct, *construction*, *finite* and *chooses* are not used quite correctly (but the observation by Osborne & Rubinstein, 1994, given below, is more unfortunate). None of these infelicities are present in Aumann's elegant proof – and claims; they are minimal.

[13] To be fair to Osborne & Rubinstein, this particular assertion *echoes* the last two sentences in Zermelo (1913; see also Larson, *op.cit*., p. 263, third full paragraph):

> 'The question as to whether the initial position p_0 already is a 'winning position for one of the players is still open. Its precise answer would of course deprive chess of its character as a game.'

That it cannot be given a 'precise answer' is the key result of Brouwerian intuitionistic constructive mathematics and recursion theory (as clarified by *Harrop's Theorem*). It is given a 'precise answer' in classical game theory only because mathematical induction – or, what Brouwer would refer to as '*complete induction*' – is unreservedly used in all known proofs of *Zermelo's Theorem on Chess* (with the notable exception, of course, of Euwe). Brouwer's negative answer – akin to *Harrop's Theorem* – relies on *Bar Induction* (see below and also Dresden, 1924 – especially pp. 37-40).

on a machine using it *trivial*[14]. Secondly, the notion of an *algorithm* deployed in this paragraph does *not* conform to the formal definition, under the *Church-Turing Thesis* and, *a fortiori*, not to any version of a *constructive algorithm*. Thirdly, there has, so far, not been an *impossibility proof* – using any notion of proof – for the use of such an 'algorithm', even granting it to be one. Finally, the notion of '*solve*' used above, as in '*solvability*', has a precise definition in recursion theory – but this is not the sense in which it is used here[15].

We now know that Zermelo's proof of his Theorem was incomplete in several precise ways. We also know, as made crystal clear by Larson (*ibid*) that many of the mathematical infelicities in Zermelo were adequately settled by König, including some additional reflections by Zermelo, as a result of being prompted by König[16].

Most importantly, the repeated claims, on the *finiteness* of chess (and related parlor games), is misleading. Formal, classical, *proofs* of *Zermelo's Theorem on Chess* always embed the *finite sequences* in *a countable infinite sequence* of the natural numbers. This becomes most clear in König's proof, using what is now called the *König Lemma*[17] (cf., Larson, ibid, p. 263, which is masterly, but Franchella, 1997[18] is mathematically – and, historically – much more complete). Underpinning any use of mathematical induction – whether in the 'applicable' form of 'backwards induction' or 'the standard procedure of "dynamic programming" (Hart, op.cit., p. 31, Remark 6.3 and footnote 20) – is an appeal to the *Axiom of Choice* (or some form of Zermelo's *well-ordering theorem*)[19].

[14] Proving the existence of, say, a Walras-Arrow-Debreu equilibrium is, at least these days, a 'trivial' exercise; but computing the equilibrium that is proved non-constructively or uncomputably is definitely not trivial, as any acquaintance with Scarf's variety of computable general equilibrium would show (cf., for example, Scarf, 1973).

[15] The use of the word 'calculated' in the quoted paragraph is gratuitous.

[16] Just for the record, von Neumann was fully aware of all this – even long before he became an adherent of *Hilbert's Program*.

[17] *König's Lemma* is also referred to as the *Unendlichkeitslemma* (Dummett, 1977, p. 49).

[18] My own acquaintance with Franchella (1997) is entirely due to Larson's reference to it.

[19] Or, in the case of dynamic programming, some other non-constructive or uncomputable assumption on the space over which an optimal policy is derived.

In the case of *König's Lemma* – as in the case of the proof by induction by Aumann or Hart – it is not always easy to detect the exact place where these non-constructive or uncomputable steps are invoked. For example, the *Axiom of Choice* is fully exploited in the proof of *König's Lemma* (cf., Franchella, ibid, p. 30, proof of Theorem E), but where it need not be invoked.

Before concluding this section, and as a preparatory vision for the final section, it may well be useful to summarise the gist of the whole approach taken in this paper with Dummett's detailed observation on the intuitively constructive and effective status of *König's Lemma*:

> "*Intuitionistically*, the .. proof of König's Lemma is invalid … . Moreover, we cannot remedy the situation by modifying the proof of König's Lemma, since reflection on the difficulty involved shows that there is no reason to suppose König's Lemma to be *constructively* true. Intuitionistically understood, the assertion that *there exists* an infinite path amounts to the claim that we can *effectively* define such a path; but the mere fact that there is no finite upper bound on the lengths of paths does not supply us with any way of doing this, since we have no *effective* means of deciding, for each given node, whether or not it is the case that there is a finite upper bound on the lengths of paths going through it." Dummett (2000), p. 49; italics added[20].

It is, therefore, misleading to suggest that Zermelo, in deriving his result on Chess provided an 'algorithm'; nor is it very fruitful to claim, in any proof of *Zermelo's Theorem on Chess* using mathematical (or, what Brouwer would refer to as *Complete*) induction, provides a method for calculating anything *precisely*, or even with *approximate precision*[21].

§ 3. Towards New Frontiers of Perfect Information Games

> "*The axiom of determinateness* is quite a strange axiom. … [It] does not appear to be an axiom of set theory. An ordinary game terminates within a *finite time*, while the axiom of determinateness refers to games which take *infinite time* to be played. The axiom of determinateness states that, in such a game, *one of the players had a winning strategy*." Takeuti, 2003, p. 71; italics added

[20] The italicized words are to be understood, and interpreted, in precise constructive and recursion theoretic senses. Most importantly, I have italicized the phrase 'there exists' because this is an essential non-constructive phrase, especially in the context of König's Lemma.
[21] This melancholy negative observation applies to all such claims regarding sub-game perfect equilibria, as well. That particular examples can be devised to show the derivation of such equilibria – as, for example, in Hart (*op.cit*., p. 32 and figure 6.4, p.31) does not make any dent in this infeasibility result in constructive mathematics (cf. Dresden, *op. cit*, pp. 39-40) and recursion theory (cf. *Harrop's Theorem*, above).

There are, to the best of this author's knowledge, two ways to algorithmize games of perfect information. One is to follow the path-breaking work of Rabin (1957); the other is via constructive mathematics. In the latter case, my own advocacy is of Brouwerian Intuitively constructive mathematics. Both avenues eschew traditional set theory and the orthodox framework of **ZFC**.

The former is generally referred to as *Arithmetic Games* or *Alternating Games*[22] and are a well-developed topic at the frontiers of aspects of recursion theory. The latter is less developed, but due to Euwe (1929) was much earlier alternative to the standard set theoretical approach of ordinary game theory.

In both cases an aim is to go beyond – or avoid – any appeal to the *Axiom of Choice*, which is intrinsically non-constructive and non-algorithmic. In connection with the introduction of the *Axiom of Determinacy*, to replace the *Axiom of Choice*, in the context of **AG**, the considered reflection of Steinhaus (1965, p. 465; italics in the original) is worth remembering:

> "All these considerations [of the conundrums arising in Banach-Mazur **AG**] impelled me to place the blame on the Axiom of Choice. Sixty years[23] of the theory of sets have elapsed since this Axiom was proclaimed, some ideas like those in Section 1 of this talk have convinced me that a purely negative attitude against [the Axiom of Choice] would be dangerous to propose. Thus I have chosen the idea of replacing [the Axiom of Choice] by the following 'Axiom of Determinacy' (*AD*):
>
> 'All two-person alternating games with perfect information are closed.'
>
> …..
>
> Axiom *AD* and its consequences have been studied by Jan Mycielski (1964). He mentions there the equivalence of Cauchy's definition of continuity with that given by Heine: it results from the usual system of axioms (i.e., the Zermelo-Fraenkel-Skolem system) with *AD* but without [the Axiom of Choice]."

Similarly, if one would wish to develop the Euwe approach, instead of working within the framework of **AG**, without appealing either to *complete induction* or the *Axiom of Choice*, which is my own preferred alternative[24], then the framework offered by Brouwer's intuitionistically constructive mathematics seems the best way forward. In this case, constructive Zermelo-

[22] I shall use **AG**, which is meant to refer to either Arithmetic or Alternating Games.
[23] The paper by Steinhaus was published in 1965.
[24] Although till now I have worked to develop AG, beginning with Velupillai (2000).

Fraenkel Set theory, with *Bar Induction* may enable one to make some advances on the path Euwe opened for us. This means the triptych of *sets*, *spreads* and *proof* have to be encapsulated within the Brouwerian intuitionistic constructive framework provided by the *Principle of Continuous Choice*, the *Fan Theorem* and *Bar Induction*.

This entails a wholly new approach to mathematical game theory. We will have to stand on the shoulders of Brouwer – and Euwe.

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