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COMPUTABILITY AND ALGORITHMIC COMPLEXITY IN ECONOMICS✣

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✣ Rabin's effectivization of the Gale-Stewart Game [42] remains the model methodological contribution to the field for which Velupillai coined the name *Computable Economics* more than 20 years ago. Alain Lewis was the first to link Rabin's work with Simon's fertile concept of *bounded rationality* and interpret them in terms of Alan Turing's work. Solomonoff (1964), one of the three -- the other two being Kolmogorov and Chaitin -- acknowledged pioneers of algorithmic complexity theory, had his starting point in one aspect of what Velupillai [72] came to call the *Modern Theory of Induction*, an aspect which had its origins in Keynes [23]. Kolmogorov's resurrection of von Mises [80] and the genesis of Kolmogorov complexity via computability theoretic foundations for a frequency theory of probability has given a new lease of life to finance theory [49]. Rabin's classic of computable economics stands in the long and distinguished tradition of game theory that goes back to Zermelo [84], Banach & Mazur [5], Steinhaus [62] and Euwe [14].

Abstract

This is an outline of the origins and development of the way computability theory and algorithmic complexity theory were incorporated into economic and finance theories. We try to place, in the context of the development of *computable economics*, some of the classics of the subject as well as those that have, from time to time, been credited with having contributed to the advancement of the field. Speculative thoughts on where the frontiers of computable economics are, and how to move towards them, conclude the paper. In a precise sense – both historically and analytically – it would *not* be an exaggeration to claim that both the origins of computable economics and its frontiers are defined by two classics, both by Banach and Mazur: that one page masterpiece by Banach and Mazur ([5]), built on the foundations of Turing's own classic, and the unpublished Mazur conjecture of 1928, and its unpublished proof by Banach ([38], ch. 6 & [68], ch. 1, §.6). For the undisputed original classic of computable economics is Rabin's *effectivization* of the Gale-Stewart game ([42]; [16]); the frontiers, as I see them, are defined by recursive analysis and constructive mathematics, underpinning computability over the computable *and constructive* reals and providing computable foundations for the economist's Marshallian penchant for *curve-sketching* ([9]; [19]; and, in general, the contents of **Theoretical Computer Science**, Vol. 219, Issue 1-2). The former work has its roots in the Banach-Mazur game (cf. [38], especially p.30), at least in one reading of it; the latter in ([5]), as well as other, earlier, contributions, not least by Brouwer.

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1 A Setting for Computability Theory and Algorithmic Complexity Theory in Economics

> "M.O. Rabin .... was the first ... to make a significant application of recursion theory to the theory of games. In Rabin ([42]) it is remarked that 'It is obvious that not all games that are considered within the theory of games are actually playable by human beings.'[1] Here we find H. Simon's [[51]] concept of bounded rationality as a hidden theme, for the point of Rabin's inquiry is to determine if certain games of the Gale-Stewart variety can be won consistently by Turing Machines that serve as surrogate players. To quote Rabin [[42], p. 147] once more: 'The question arises as to what extent the existence of winning strategies makes a win-lose [i.e., zero-sum] game trivial to play. Is it always possible to build a computer which will play the game and consistently win?'
>
> What Rabin is doing here is to provide an interpretation of Simon's concept of bounded rationality that is computational in character. The significance of [[42]] is that the techniques of recursion theory are used to fix a precise interpretation of computability within Church's Thesis."
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> [27], p. 84; underlining in the original.

Alain Lewis is a contemporary pioneer, whose research program on *Effectively Constructive Mathematics* ([28], [27]) had an immense flowering in the years between the mid-1980s and the early 1990s. In his remarkably prescient *Monograph* (manuscript), [27], the above elegant interpretation of (what he calls) *Rabin's Theorem* ([42]), brings together the three undisputed pioneers of computable economics, i.e., *Herbert Simon*, *Michael Rabin* and *Alain Lewis* himself, in one fell swoop, so to speak.

Velupillai's earliest attempt at considering economic theoretical issues in terms of computability theory goes back to his work on the computational complexity of algorithms for mathematical programming formalizations of optimization models in macroeconomics, from about 1976 – when Alan Turing, had not his life been cut short by tragic events, would have been 64 years of age, the age Velupillai has now reached!

Now, 36 years later, we commemorate the Turing Centennial with a well established field of Computable Economics fully cognizant of the pioneering work of Alan Turing and its relevance to many aspects of economic thoery, applied economics, human problem solving and much else. It will not be incongruous or inappropriate in any way if we single out two pioneering economists, Herbert Simon and Alain Lewis, as those most responsible for contributing the initial

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[1] The exact quotation is ([42], p. 147):

   "It is quite obvious that not all games which are considered in the theory of games can actually be played by human beings."

impulse towards what came to be Computable Economics – in spite of earlier stirrings by philosophers (Hilary Putnam) and computer scientists (Michael Rabin).

This 'next step in [economic] analysis', conjectured the *doyen* of mathematical economics, Kenneth Arrow ([1], p.S398), ' [would be] a more consistent assumption of *computability* in the formulation of economic hypotheses'. But this has *not* been taken by economic theorists or, more pertinently, by anyone claiming to be a *computational* economist, *computable* general equilibrium theorist, applied *computable* general equilibrium theorist, *algorithmic* game theorist, so-called agent-based economic and financial modeller or any variety of *DSGE*[2] theorist. Indeed, not too long after the famous, and decidedly *non-computable* and *non-constructive*, *Arrow-Debreu classic* was published ([2], the trio of outstanding mathematical economists, Arrow, Karlin and Scarf, cautioned economists against facile conflation of *existence theorems* and *effectively computable solutions* ([3], p.17). Despite this early 'warning' by three of the pioneering mathematical economists of economic theorising in the non-computable mode, only in the sadly aborted research program on *effectively constructive economics* by Alain Lewis and in my *computable economics,* have there been *systematic* and coherent attempts to take Arrow's conjecture seriously[3]. As far as I am concerned, Simon ([51]), together with Michael Rabin ([42][4]) and Alain Lewis, are the undisputed pioneers of *Computable Economics*, and both of these classics appeared in the public domain before ([3]). In [72] it was pointed out that (pp. 25-6):

> "[Simon's] path towards a broader base for economics .... stressed two empirical facts (quotes are from [53], p. x):
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> (I). 'There exists a basic repertory of mechanisms and processes that Thinking Man uses in all the domain in which he exhibits intelligent behavior.';
>
> (II). 'The models we build initially for the several domains must all be assembled from this same basic repertory, and *common principles of architecture* must be followed throughout.' (italics added);"
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> It is at this point that I feel Simon's research program pointed the way toward computable economics in a precise sense. .....
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> Instead, the direction Simon took codified his research program in

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[2] **D**ynamic **S**tochastic **G**eneral **E**quilibrium.

[3] *Computable Economics* is a name I coined in the early 1980s, from the outset with the intention of encapsulating computability and constructivity assumption in economic theory. My earliest recollection is 1983, when I announced a series of graduate lectures on *Turing and his Machine for Economists*, in the department of economics at the European University Institute. Only one person signed up for the course, Henrietta Grant-Peterkin, one of our valued departmental secretaries! The course was still-born.

[4] In one of the most elegantly written 'eternal' classics of recursion theory, Hartley Rogers ([44]), the one blemish I found is the relegation of Rabin's results to a minor problem (p.121, ex. 8.5), with the unfortunate comment: 'This is a special and trivial instance of a general theorem about games'!

terms of the familiar notions of bounded rationality and satisficing [underpinned by computational complexity theory] ..

I remain convinced that, had Simon made the explicit recursion-theoretic link at some point in the development of his research program, computable economics would have been codified much earlier."

After reading [72], Simon wrote Velupillai as follows (italics added):

" As the book makes clear, my own journey through bounded rationality *has taken a somewhat different path*. Let me put it this way. There are many levels of *complexity in problems*, and corresponding boundaries between them. *Turing computability is an outer boundary*, and as you show, any theory that requires more power than that surely is irrelevant to any useful definition of human rationality. ....

Finally, we get to the empirical boundary, measured by laboratory experiments on humans and by observation, of the level of complexity that humans actually can handle, with and without their computers, and - perhaps more important – what they actually do to solve problems that lie beyond this strict boundary even though they are within some of the broader limits.

The latter is an important point for economics, because we humans spend most of our lives making decisions that are far beyond any of the levels of complexity we can handle exactly; and this is where satisficing, floating aspiration levels, recognition and heuristic search, and similar devices for arriving at good-enough decisions take over. A parsimonious economic theory, and an empirically verifiable one, shows how human beings, using very simple procedures, reach decisions that lie far beyond their capacity for finding exact solutions by the usual maximizing criteria "

Simon chose to work within the '*empirical* boundary', recognising immediately that computable economics was an attempt at defining, effectively, the relevance of the 'outer boundary' for formalisation in economic theory.

The true significance of Lewis's insight was to realise that Simon's concept of bounded rationality had to be given computational content; that Lewis did not also realise that Simon did give it this content from the outset is besides the point. But to give the notion of bounded rationality computational content in the context of games played by computing machines is one thing; to interpret bounded rationality as encapsulated in *finite automata* is quite another thing. Fortunately, Lewis did not fall into the latter trap, one which many distinguished game theorists almost willingly embraced.

The point missed by Lewis in his handsome tribute to Rabin is that this classic came down in the great tradition of *alternating games* (see [71]), begun by Zermelo at the beginning in ([84]), on the one hand; and, on the other hand, down the even nobler and more ancient tradition of what is now called

combinatorial games (see the recent elegant, and eminently readable, [35] for a fine exposition of the history and origins of this field, with copious references). But there are many eminent game theorists who feel able to claim Zermelo as a precursor of orthodox game theory. In some senses – particularly with regard to von Neumann's original min-max result and to the sustained non-constructive and uncomputable methodology that underpins formal, orthodox, game theory - this claim many have a modicum of truth to it.

Our own 'take' on Rabin's classic as the fountainhead of computable economics is its pedagogic value in providing a tutorial on how to effectivise a non-effective framework in orthodox theory – whether economic or game theoretic. This is what I have emphasised in [71]. But, of course, it has also led to a revitalisation of both a part of recursion theory (see p. 254 in the excellent – although slightly dated – survey by Telgársky, [64], of recursion theoretic work inspired by Banach-Mazur games for some of the early and classic references), and a reflection on the possibility of avoiding reliance on the *axiom of choice* (see below, the comment on the *axiom of determinacy*).

The von Neumann paper of 1928 ([81]), the 'official' fountainhead for *orthodox game theory*, etched indelibly, to an essentially non-existent Mathematical Economics community, what has eventually come to be called '*Hilbert's Dogma*'[5], 'consistency $\Leftrightarrow$ existence'. This became – and largely remains – the mathematical economist's *credo*. Hence, too, the inevitable schizophrenia of 'proving' existence of equilibria, first, and looking for methods to *construct* and *compute* them at a second, entirely unconnected, stage. Thus, too, the indiscriminate appeals to the *tertium non datur* – and its implications – in 'existence proofs', on the one hand, and the ignorance about the nature and foundations of constructive mathematics and computability theory, on the other.

But it was not as if von Neumann was *not* aware of Brouwer's opposition to 'Hilbert's Dogma', even as early as 1928, although there is reason to suspect that something peculiar may have been going on. Hugo Steinhaus observed, ([62]):

> "[My] inability [to prove the minimax theorem] was a consequence of the ignorance of Zermelo's paper in spite of its having been published in 1913. .... J von Neumann was aware of the importance of the minimax principle [in [81]]; it is, however, *difficult to understand the absence of a quotation of Zermelo's lecture in his publications.*"
>
> ibid, p. 460; italics added

Why didn't von Neumann refer, in 1928, to the Zermelo-tradition of (*alternating*) *games*? van Dalen, in his comprehensive and scrupulously fair biogra-

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[5] In van Dalen's measured, scholarly, opinion, [69], pp. 576-7 (italics added):

"Since Hilbert's yardstick was calibrated by the continuum hypothesis, *Hilbert's dogma, 'consistency $\Leftrightarrow$ existence'*, and the like, he was by definition right. *But if one is willing to allow other yardsticks*, no less significant, but based on alternative principles, then Brouwer's work could not be written off as obsolete nineteenth century stuff."

phy of Brouwer, [69], p. 636, noted (italics added), without additional comment that:

> "In 1929 there was another publication in the intuitionistic tradition: an intuitionistic analysis of the game of chess by Max Euwe[6]. It was a paper in which the game was viewed as a spread (i.e., a tree with the various positions as nodes). Euwe carried out *precise constructive estimates* of various classes of games, and considered the influence of the rules for draws. When he wrote his paper he was not aware of the earlier literature of Zermelo and Dénès König. *Von Neumann called his attention to these papers, and in a letter to Brouwer, von Neumann sketched a classical approach to the mathematics of chess, pointing out that it could easily be constructivized.*"

von Neumann dinn't provide this 'easily constructivized' approach – then, or later? Perhaps it was easier to derive propositions appealing to the *tertium non datur,* and to 'Hilbert's Dogma', than to do the hard work of *constructing estimates* of an algorithmic solution*, as* Euwe did[7]? Perhaps it was easier to continue using the *axiom of choice* than to construct new axioms – say the *axiom of determinacy*[8] – as Steinhaus and Mycielski did ([32])? Whatever the reason, the fact remains that the von Neumann legacy was a legitimization of 'Hilbert's Dogma' and the indiscriminate use of the axiom of choice in mathematical economics and game theory.

Velupillai began to think of Game Theory in algorithmic modes – but not what is today referred to as *Algorithmic Game Theory* – after realizing the futility of algorithmising the uncompromisingly *subjective* von Neumann-Nash

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[6] In a strange lapse, van Dalen refers to Euwe, 1929, without giving the exact details of the reference in his excellent bibliography. The exact reference is [14]. Max Euwe was the fifth World Chess Champion, between 1935-1937, having defeated Alexander Alekhine, on December 15, 1935.

[7] At the end of his paper Euwe reports that von Neumann brought to his attention the works by Zermelo and König, after he had completed his own work (ibid, p. 641). This further substantiates the perplexity reported by Steinhaus (above) on the absence of any reference to Zermelo in von Neumann's official publications of the time. In any case, Euwe then goes on (italics added):

> "*Der gegebene Beweis is aber nicht konstruktive*, d.h. es wird keine Methode angezeigt, mit Hilfe deren der gewinnweg, wenn überhaupt möglich, *in endlicher Zeit konstruiert werden kann.*"

[8] The introduction of this axiom is relevant in computable economics and point we wish to make is best described in Takeuti's observation ([63], pp. 73-4; italics added):

> "There has been an idea, which was originally claimed by Gödel and others, that, if one added an axiom which is a strengthened version of the existence of a measurable cardinal to existing axiomatic set theory, then various mathematical problems might all be resolved. Theoretically, nobody would oppose such an idea, but, in reality, *most set theorists felt it was a fairy tale and it would never really happen.* But it has been *realized by virtue of the axiom of determinateness,* which showed Gödel's idea valid."

approach to game theory and beginning to understand the importance of *Harrop's theorem* ([18]). This realization came after an understanding of *effective playability* in arithmetical games, developed elegantly by Michael Rabin.

The brief, rich and primarily recursion theoretic framework of Harrop's classic paper requires a deep understanding of the rich interplay between *recursivity and constructive representations* of **sets** that are *recursively enumerable.* There is also an obvious and formal connection between the notion of a finite combinatorial object, whose complexity is formally defined by the uncomputable Kolmogorov measure of complexity, and the results in Harrop's equally pioneering attempt to characterise the recursivity of finite sets and the resulting indeterminacy – undecidability – of a Nash equilibrium *even in the finite case.* To the best of my knowledge this interplay has never been mentioned or analysed in the mathematical economic or game theoretic literature.

When Velupillai conceived the notion of computable economics in the early 1980s, he had in mind both constructive and computable mathematics as bases for the formalization of economic theory. He was blissfully ignorant of the pioneering works by Rabin and Lewis, till about the late 1980s. Also, the important work by Douglas Bridges based on constructive mathematics were unknown to him when he was fashioning computable economics including constructive assumptions and interpretations.

Finally, anyone even remotely familiar with Conway's characteristically clear note on *A Gamut of Game Theories* ([12]) and Turing's classic on *Solvable and Unsolvable Problems* ([67]), and Herbert Simon's kind of behavioural economics – called classical behavioural economics in this paper – will know that there is an almost formal duality between problem solving and (combinatorial) games. This is not a theme space allows us to develop, but it needs to be pointed out that any future for computable economics will have to enlarge on this aspect of the interaction between recursion theory, combinatorial games, *Ramsey theory* and behavioural economics.

The paper is organised as follows. The next section is a retrospective of some of the results obtained under the rubric of computable economics. The section is sub-divided into two sub-sections: classical behavioural and (classical) computable economics. Section is a view of randomness and (statistical) induction, underpinned by algorithmic complexity theory, but with suggestions on an unususal double duality: one between algorithmic complexity theory and computational complexity theory; the other between classical recursion theory and constructive analysis. The final section outlines aspects of our view of the frontiers of computable economics. The main vision here is the hope that 'the next step in computable economic analysis would be a more consistent' consideration of recursive or computable analysis, particularly in macroeconomic dynamics.

2 Computability in Economics: A Retrospective

> "[The] adoption of the *infinitary, nonconstructive, set theoretic*, algebraic, and structural methods that are characteristic to modern mathematics [. . . .] were controversial, however. At issue was not just whether they are consistent, but, more pointedly, whether they are meaningful and appropriate to mathematics. After all, *if one views mathematics as an essentially computational science*, then arguments without computational content, whatever their heuristic value, are not properly mathematical. .. [At] the bare minimum, we wish to know that the universal assertions we derive in the system will not be contradicted by our experiences, and *the existential predictions will be borne out by calculation*. This is exactly what *Hilbert's program*[9] was designed to do."
>
> [4], pp. 64-5; italics added

Thus, our claim is that the *existential predictions* made by the purely theoretical part of mathematical economics, game theory and economic theory '*will* [*not*] *be borne out by calculations.'* There is, therefore, a serious *epistemological deficit* – in the sense of economically relevant knowledge that can be processed and accessed computationally and experimentally – in all of the above approaches, claims to the contrary notwithstanding, that is unrectifiable without *wholly abandoning their current mathematical foundations*. This is an *epistemological deficit* even before considering the interaction between appeals to infinite – even uncountably infinite – methods and processes in proofs, where both the universal and existential quantifiers are freely used in such contexts, and the *finite* numerical instances with which they are, ostensibly, 'justified'. This epistemological deficit requires even 'deeper' mathematical and philosophical considerations in *Cantor's Paradise*[10] of ordinals[11] , where combinatorics, too, have to be added to computable and constructive worlds to make sense of

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[9] Velupillai has tried to make the case for interpreting the philosophy and methodology of mathematical economics and economic theory in terms of the discipline of *Hilbert's program* in [?].

[10] Hilbert did not want to be driven out of '*Cantor's Paradise*' ([21]; p.191):

> 'No one shall drive us out of the paradise which Cantor has created for us.'

To which the brilliant 'Brouwerian' response, if we may be forgiven for stating it this way, by Wittgenstein was ([83]; p.103):

> 'I would say, "I wouldn't dream of trying to drive anyone out of this paradise." I would try to do something quite different: I would try to show you that it is not a paradise – so that you'll leave of your own accord. I would say, You're welcome to this; just look about you." '

[11] Where '*Ramsey Theory*', '*Goodstein Sequences*' and the '*Goodstein theorem*', reign supreme. In work in progress these issues are dealt with in some detail, as they pertain to bridging the '*epistemological deficit*' in economic theoretical discourse in the mathematical mode.

claims by various mathematical economists and agent based modeling practitioners.

Against this backdrop, within the framework of what we will now call *classical* computable economics, the following are some of the results that have been derived[12]: (1). Nash equilibria of (*even*) *finite* games are constructively indeterminate; (2). The Arrow-Debreu equilibrium is uncomputable (and its existence is proved nonconstructively); (3). The Uzawa Equivalence Theorem is uncomputable and nonconstructive; (4). Computable General Equilibria are neither computable nor constructive; (5). The Two *Fundamental Theorems of Welfare Economics* are Uncomputable and Nonconstructive, respectively; (6). The *Negishi method* is proved nonconstructively and the implied procedure in the method is uncomputable; (7). There is no effective procedure to generate preference orderings; (8). Rational expectations equilibria are uncomputable and are generated by uncomputable and nonconstructive processes; (9). Policy rules in macroeconomic models are noneffective; (10). Recursive Competitive Equilibria (RCE), underpinning the Real Business Cycle (RBC) model and, hence, the *Dynamic Stochastic General Equilibrium* (DSGE) benchmark model of Macroeconomics, are uncomputable; (11). Dynamical systems underpinning growth theories are incapable of computation universality; (12). There are games in which the player who in theory can always win cannot do so in practice because it is impossible to supply him with effective instructions regarding how he/she should play in order to win; (13). The theoretical benchmarks of Algorithmic Game Theory are uncomputable and non-constructive; (14). Boundedly rational agents,satisfying, formalised within the framework of (metamathematical) decision problems are capable of effective procedures of rational choice.

In the next subsection we outline the computability theoretic background against which # 14 can be demonstrated. The second subsection is a brief outline of classical computable economics, in retrospective mode.

2.1   Notes on *Classical* Behavioural Economics - Computable Foundations

"If we hurry , we can catch up to Turing on the path he pointed out
to us so many years ago."

Herbert Simon, [55], p. 101.

Velupillai coined the phrase *classical behavioural economics* to characterise the kind of behavioural economics pioneered by Herbert Simon, which was underpinned, at every level of theoretical and applied analysis, by a model of computation. Invariably, although not always explicitly, it was *Turing's model of computation*. To highlight the difference between *modern* behavioural economics, which is *never* underpinned by *a model of computation*, and the kind of behavioural economics that was pioneered and practiced by Simon and his

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[12]Apart from the twelfth result, which is due to the pioneering work of Michael Rabin ([42]) in 1957, the rest are due to Velupillai. The first was suggested by Francisco Doria.

associates and followers, Velupillai decided to refer to the latter as practitioners of *classical behavioural economics*[13].

The fundamental focus in classical behavioural economics is on *decision problems* faced by *human problem solvers*, the latter viewed as information processing systems. All of these terms are given computational content, *ab initio*. But given the scope of this paper we shall not have the possibility of a full characterisation. The ensuing 'bird's eye' view must suffice for now[14].

A *decision problem* asks whether there exists an *algorithm* to *decide* whether a mathematical assertion does or does not have a proof; or a formal problem does or does not have an algorithmic solution. Thus the characterization makes clear the crucial role of an underpinning *model of computation*; secondly, the answer is in the form of a *yes/no* response. Of course, there is the third alternative of '*undecidable*', too. It is in this sense of *decision problems* that we interpret the word 'decisions' here.

As for 'problem solving', we shall assume that this is to be interpreted in the sense in which it is defined and used in the monumental classic by Newell and Simon ([33]).

Finally, the *model of computation* is the *Turing model*, subject to the *Church-Turing Thesis*.

To give a rigorous mathematical foundation for bounded rationality and satisficing, as decision problems[15], it is necessary to underpin them in a dynamic model of choice in a computable framework. However, any formalization underpinned by a model of computation in the sense of computability theory is intrinsically dynamic.

Consider the Boolean formula:

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \{\neg x_2\}) \wedge (x_2 \vee \{\neg x_3\}) \wedge (x_3 \vee \{\neg x_1\}) \wedge (\{\neg x_1 \vee \{\neg x_2\} \vee \{\neg x_3\})$$
$$(1)$$

Remark 1 *Each subformula within parenthesis is called a clause; The variables and their negations that constitute clauses are called literals; It is 'easy' to 'see' that for the truth value of the above Boolean formula to be $t(x_i) = 1$, all the*

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[13] See [48] for a more detailed discussion of this theme.

[14] Some details are discussed in greater and more rigorous depth in [73]

[15] The three most important classes of decision problems that almost characterise the subject of computational complexity theory, underpinned by a model of computation – in general, the model of computation in this context is the *Nondeterministic Turing Machine* – are the **P**, **NP** and **NP-Complete** classes. Concisely, but not quite precisely, they can be described as follows:

1. **P** defines the class of computable problems that are solvable in time bounded by a *polynomial function of the size of the input*;

2. **NP** is the class of computable problems for which a *solution* can be *verified in polynomial time*;

3. A computable problem lies in the class called **NP-Complete** if every problem that is in **NP** can be *reduced to it in polynomial time*.

subformulas within each of the parenthesis will have to be true. It is equally 'easy' to see that no truth assignments whatsoever can satisfy the formula such that its global value is true. This Boolean formula is unsatisfiable.

Problem 2 *SAT – The Satisfiability Problem*

Given m clauses, $C_i(i = 1, \ldots, m)$, containing the literals (of) $x_j(j = 1, \ldots, n)$, *determine* if the formula $C_1 \wedge C_2 \wedge \ldots \ldots \wedge C_m$ is *satisfiable*.

Determine means 'find an (efficient) algorithm'. To date it is not known whether there is an *efficient* algorithm to solve *the satisfiability problem* – i.e., to determine the truth value of a Boolean formula. In other words, it is not known whether $SAT \in \mathbf{P}$. But:

Theorem 3 $SAT \in \mathbf{NP}$

Finally, we have Cook's famous theorem:

Theorem 4 *Cook's Theorem*
   SAT is $\mathbf{NP}$ *– Complete*

It is in the above kind of context and framework within which we are interpreting Simon's vision of behavioural economics. In this framework optimization is a very special case of the more general decision problem approach. The real mathematical content of *satisficing*[16] is best interpreted in terms of the satisfiability problem of computational complexity theory, the framework used by Simon consistently and persistently - and a framework to which he himself made pioneering contributions.

Finally, there is the computably underpinned definition of *bounded rationality*.

Theorem 5   *The process of rational choice – i.e., boundedly rational choice – by an economic agent is formally equivalent to the computing activity of a suitably programmed (Universal) Turing machine.*

Proof.   By construction. See §3.2, pp. 29-36, *Computable Economics* [[72]] ∎

Remark 6 *The important caveat is 'process' of rational choice, which Simon tirelessly emphasized by characterizing the difference between 'procedural' and 'substantive' rationality; the latter being the defining basis for Olympian rationality ([54], p.19), the former that of the computationally underpinned problem solver facing decision problems. In the Olympian model the 'process' aspect is*

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[16]In [56], p. 295, Simon clarified the *semantic* sense of the word *satisfice*:

"The term 'satisfice', which appears in the *Oxford English Dictionary* as a Northumbrian synonym for 'satisfy', was borrowed for this new use by H. A. Simon (1956) in 'Rational Choice and the Structure of the Environment' [i.e, [52]]".

submerged and dominated by the static optimization operator. By transforming the agent into a problem solver, constrained by computational formalisms to determine a decision problem, Simon was able to extract the procedural content in any rational choice.

Definition 7 *Computation Universality of a Dynamical System*

A dynamical system is said to be capable of computation universality if, using its initial conditions, it can be programmed to simulate the activities of any arbitrary Turing Machine, in particular, the activities of a Universal Turing Machine.

Theorem 8 *Boundedly rational choice by an information processing agent within the framework of a decision problem is capable of computation universality.*

Proof. See [74]. ■

We have only scratched a tiny part of the surface of the vast canvass on which Simon sketched his vision of a computably underpinned behavioural economics. Nothing in Simon's behavioural economics – i.e., in *Classical Behavioural Economics* – was devoid of computable content. There was – is – never any *epistemological deficit* in any computational sense in classical behavioural economics.

2.2   *Classical* Computable Economics

"The method of 'postulating' what we want has many advantages; they are the same as the advantages of theft over honest toil. Let us leave them to others and proceed with our honest toil."

Bertrand Russell ([46], p. 71)

In computable economics, as in any computation with analogue computing machines or in classical behavioural economics, all solutions are based on *effectively computable* methods[17]. Thus computation is intrinsic to the subject and all formally defined entities in computable economics – as in classical behavioural economics – are, therefore, algorithmically grounded. Given the algorithmic foundations of computability theory and the intrinsic dynamic form and content of algorithms, it is clear that this will be a '*mathematics with dynamic and algorithmic overtones*'[18]. This means, thus, that computable economics is

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[17] We identify five varieties of computation, underpinned by computability theory, even if not explicitly: classical recursion theory, computable analysis, constructive analysis, interval analysis and classical numerical analysis (now given computable foundations in [7]).

[18]

"I think it is fair to say that for the main existence problems in the theory of economic equilibrium, one can now bypass the fixed point approach and attack the equations directly to give existence of solutions, with a simpler kind of mathematics and *even mathematics with dynamic and algorithmic overtones*."
[58], p.290; italics added.

a case of a new kind of mathematics in old economic bottles. The 'new kind of mathematics' implies new questions, new frameworks, new proof techniques - all of them with algorithmic and dynamic content for digital domains and ranges.

Some of the key formal concepts of computable economics are, therefore: *solvability & Diophantine decision problems, decidability & undecidability, computability & uncomputability, satisfiability, completeness & incompleteness, recursivity* and *recursive enumerability, degrees of solvability (Turing degrees), universality* & the *Universal Turing Machine* and *Computational, algorithmic* and *stochastic complexity.* The proof techniques of computable economics, as a result of the new formalisms, will be, typically, invoking methods of: *Diagonalization, The Halting Problem for Turing Machines, Rice's Theorem, Incompressibility theorems, Specker's Theorem, Recursion Theorems.* For example, the *recursion theorems* will replace the use of traditional, non-constructive and uncomputable, topological fix point theorems, routinely used in orthodox mathematical analysis. The other theorems have no counterpart in non-algorithmic mathematics.

In the spirit of pouring new mathematical wines into old economic bottles, the kind of economic problems that computable economics is immediately able to grant a new lease of life are the classic ones of: computable and constructive existence and learning of rational expectations equilibria, computable learning and complexity of learning, computable and bounded rationality, computability, constructivity and complexity of general equilibrium models, undecidability, self-reproduction and self-reconstruction of models of economic dynamics (growth & cycles), uncomputability and incompleteness in (finite and infinite) game theory and of Nash Equilibria,decidability (playability) of arithmetical games, the intractability (computational complexity) of optimization operators; etc.

Suppose the starting point of the computable economist whose visions of actual economic data, and its generation, are the following:

Conjecture 9 *Observable variables are sequences that are generated from recursively enumerable but not recursive sets, if rational agents underpin their generation.*

The above conjecture is is akin to the orthodox economic theorist and the econometrician assuming that all observable data emanate from a structured probability space and the problem of inference is simply to determine, by statistical or other means the parameters that characterise their probability distributions.

All the way from microeconomic supply and demand functions to monetary macroeconomic variables, parameters and functions, *Diophantine relations, equations and functions predominate in computable economics.* This is because the natural data types in economics are, at best, rational numbers. Hence, the following famous theorem is used extensively.

The following three theorems of classical computability theory ([36]), for example, are used to prove the uncomputability of rational expectations equilibria in orthodox frameworks, to construct computable rational expectations

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equilibria in computable macroeconomics and to formalise computable (macro-economic) growth theory, respectively: **Rice's Theorem**, the **Halting Problem for Turing Machines** (the recursion theoretic) **Fixed Point Theorem** and the **Recursion Theorem** (related to invariance theorems in the domain of algorithmic complexity theory).

The idea behind the *recursion theorem* is to formalize the activity of a Turing Machine that can obtain its own description and, then, compute with it. This theorem is essential, too, for formalizing, recursion theoretically, a model of *growth* in a macroeconomy and to determine and learn, computably and constructively, *rational expectations equilibria*. The *fix point theorem* and the *recursion theorem* are also indispensable in the computable formalization of *policy ineffectiveness* postulates, *time inconsistency* and *credibility* in the theory of macroeconomic policy. Even more than in microeconomics, where topological fix point theorems have been indispensable in the formalizations underpinning existence proofs, the role of the above *fix point theorem* and the related *recursion theorem* are absolutely fundamental in what I come to call *Computable Macroeconomics*.

Anyone who is able to formalize these theorems, corollaries and conjectures and work with them, would have mastered some of the key elements that form the core of the necessary mathematics of computable economics. Unlike so-called computable general equilibrium theory and its offshoots, computable economics – and *its* offshoots – are intrinsically computational and numerical.

3   Randomness, Induction and Algorithmic Complexity

> "But it will be clear that, for those who hold that the mathematical universe consists of lawlike objects *only*, *Kollektivs* are equally *impossible*."
>
> [70], p.60; italics added.

We have come round to the belief, via Solomonoff ([60], [61]), that Keynes ([**?**]) is the origin, at least from an economist interested in the foundations of statistical induction (*ibid*, p. 350), of one strand of algorithmic complexity theory. However, this is not to deny the fundamental importance of von MIses ([80]), his remarkably cogent 'manifesto', *Erst das Kollektiv, dann die Wharscheinlikhkeit*, his struggles to define a consistent notion of *Kollektivs constructively* and its eventual realisation in the computability theoretic notion of (uncomputable) *Kolmogorov complexity*. Few seem to have acknowledged the imporatnce of underpinning *Kollektivs* on the Brouwerian notion of *lawless sequences*. A close reading of [23], particularly chapter 33, would, we contend, substantiate our stance that Keynes, too, was groping for such a notion, as always before its time.

Current orthodoxy of the field of algorithmic complexity theory is elegantly and comprehensively discussed, explained and described, all the way to the

frontiers of research, in the almost encyclopaedic treatises by Li & Vitanyi ([29]), Nies ([34]) and Downey & Hirschfeldt ([13]). Velupillai has had a stab at a concise outline of the field, from the point of view of randomness *and* induction, so that learning can be studied from the point of view of algorithmic complexity (cf., [**?**], chapter 5), to which we may refer the interested for a potted survey of the field[19].

The orthodox story, in an ultra-brief nutshell, is that the origins of the field of algorithmic complexity theory lie in the work of Kolmogorov, Chaitin and Solomonoff[20], in their approaches to, respectively, the quantity of information in finite objects, program-size descriptions of the information content of a finite object and *induction*. For an economist the most interesting approach is that by Solomonoff, whose starting point, in fact, was the *Treatise on Probability, [?]* by Keynes. These original aims developed into, and linked with the earlier research on, the von Mises attempt to define a frequency theory approach to probability, randomness[21] and Bayesian estimation. All this is part of the folklore of the subject, easily gleaned from any of the indicated references, above.

In all three traditions – i.e., the Kolmogorov, Chaitin and Solomonoff – the intentions were to measure the amount of information necessary to *describe* a given, finite, binary sequence (or string). A little more precisely, the idea is as follows: given a string x, its algorithmic complexity is defined to be the shortest string y from which a Universal Turing Machine[22] can 'produce' the given x. On the other hand, in computational complexity theory – particularly as a result of adherence to 'Post's Program' – attention is *not* focused on *individual finite strings*. Instead the fundamental questions are about the computational difficulty – i.e., complexity – of recognising *sets*. Thus, the problem is about *deciding* whether a given finite string belongs to a particular set or not. It will be evident that in computational complexity theory one tries to associate a function $\tau_\wp : \mathbb{N} \to \mathbb{N}$, to a recursive set, $\wp$ such that $\wp$ is accepted by those Turing Machines, say Θ, that run in *time* $\Omega\left(\tau_\wp\left(n\right)\right)$. Therefore, one way to link algorithmic complexity theory with computational complexity theory will be to define a notion of the former that is *time-bounded* and is able to capture

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[19]Recursion theoretic *Inductive Inference* is elegantly and comprehensively treated in the second volume of Odifreddi's treatise on Classical Recursion Theory ([37], in particular, VII.5 & IX.5). Learning of rational expectations equilibria in the setting of recursively enumerable sets remains an incompletely explored field in macroeconomics – as it does in finance theory.

[20]The representative references for Chaitin, Kolmogorov and Solomonoff are, respectively, [10], [26], [60] and [61].

[21]One direct link with computational complexity theory, was stated succinctly by Compagner, [11], p.700 (italics added):

> "..[T]he mathematical description of random sequences in terms of complexity, which in algorithmic theory leads to the identification of randomness with *polynomial-time unpredictabilty*."

Once algorithmic complexity theory is viewed as the basis for a definition of finite random sequences, then it is inevitable that the emphasis will be on prediction rather than computation. Thus, the link with orthodox computational complexity theory is not as firm as the inclusion of the sobriquet 'complexity' in the title may suggest.

[22]In the Solomonoff tradition the corresponding 'universality' resides in the concept of a 'universal distribution'.

aspects of the complexity of the set $\wp$. In other words, it is necessary to add a time-bounded complexity component, as is routine in computational complexity theory, to the standard measure of algorithmic complexity. If this is done, the complexity of the finite string, x, will now be defined by the minimum of the sum of the description length and a measure of the time required to produce that x, from the given description. First steps towards such an attempt is made in [77].

Since algorithmic complexity theory is, *ab initio*, underpinned by the Turing model of computation, it is natural to define time-constrained generation of the descriptive complexity of members of *sets*. For example, as a computable economist, we construct economic theories underpinned by Turing's model of computation. Given a computable economic theory, there will be naturally definable time-bounded measures to describe the theory and, hence, immediate considerations of computational complexity of such descriptions. This is the way the complexity of solutions to ODE is studied. First, the ODE is either constructified or formalised within computability theory; then the computational complexity of the constructified or computable theoretic solution is evaluated. Thus, description is intrinsically algorithmized and the computable economist can switch betwee algorithmic complexity theory and computational complexity theory, in formal, dual, ways. However:

> "There is also *a technical sense of 'complexity'* in *logic*, variously
> known as Kolmogorov complexity, Solomonoff complexity, Chaitin
> complexity, ..., algorithmic complexity, information-theoretic com-
> plexity, and program-size complexity. The most common designa-
> tion is 'Kolmogorov complexity' .... this is probably a manifestation
> of the principle of 'Them that's got shall get,' since Kolmogorov is
> the most famous of these mathematicians[23]."

> [15]; p. 137; italics added.

Franzén's perceptive observation suggests that this whole area is really about '*a technical sense of 'complexity'* in *logic*'[24]. In this sense we would like to add another point so as to dispel popular misconceptions about correlating or juxtaposing *complexity* with *incompleteness*[25]. Many an unwary reader of Chaitin's

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[23] In their comprehensive and admirable text on this subject, Li and Vitanyi first gave the reason for subsuming all these different variations on one theme by the name 'Kolmogorov Complexity', [29], p.84:

> "Associating Kolmogorov's name with [algorithmic] complexity may also be an
> example of the 'Matthew Effect' first noted in the Gospel according to Matthew,
> 25:29-30.."

[24] However, we believe Chaitin's reference to his own pioneering work as 'algorithmic information theory', is a much better encapsulation of the contents of the field and the intentions of the pioneers. Indeed, the natural precursor is Shannon, rather than von Mises, but *Whig history* is a messy affair and straightening out historical threads is a difficult task, even in a contemporary field.

[25] Or, *simplicity* with *completeness*. I am, of course, referring to *incompleteness* in the strict metamathematical sense.

important works - and his specific program-size approach to algorithmic information theory, the incarnation of Kolmogorov complexity in Chaitin's independent work - has had a tendency to claim that incompleteness, undecidability or uncomputability propositions are only valid in so-called '*sufficiently complex*' mathematical systems. *A fortiori*, that intuitively *simple* computable systems are not computationally complex. This is simply false. Very simple formal mathematical systems are capable of generating incompleteness and undecidability propositions; just as intuitively very simple computable systems are capable of encompassing incredibly complex computational complexities, as some of the above examples have shown. Conversely, there are evidently complex systems that are provably complete and decidable; and, similarly, there exist seemingly complex functions that are capable of being computed, even primitive recursively. As one obvious and famous example illustrating *incompleteness* and *essential undecidability* in an *intuitively simple, finitely axiomatizable, simply consistent*[26] theory, one can take Robinson's Arithmetic,[43], as shown in some of the classic books of metamathematics[27], eg., [24], [25], pp. 280-1, [8], p.215,ff.

If the modern origins of computational complexity theory, via computability theory, can be found in *Hilbert's Tenth Problem*, then, equally, the proto-historic origins of algorithmic complexity theory can found, via exact approximation theory, in *Hilbert's Thirteenth Problem*[28]. Hilbert's aim in formulating the *13th Problem* – based on problems of nomography[29] - was to characterise functions in

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[26] See [25], p.287, footnote 216 and [24], p.470, Theorem 53. The part played by *simple consistency* and the analogy with Rosser's result of the essential undecidability of $\mathbb{N}$, [45], is also discussed in the relevant parts of [25]. Furthermore, despite some unfortunate misprints and unclarity, Franzén's fine exposition of the use and abuse of *Gödel's Incompleteness* theorems has a good discussion of the way the *Rosser sentence* (rather than the more famous *Gödel sentence*) is used in proving – by reference to [59] – undecidability in *Robinson's Arithmetic* ([15], pp.158-9).

[27] I cite this example also because *Robinson's Arithmetic* is sufficient to represent every recursive function. It figures in the very first, 'Introductory', pages of Odifreddi's comprehensive, yet pedagogical, textbooks of *classical recursion theory, [36]*, §I.1, p.23. There are many equivalent ways of setting out the axioms of Robinson's Arithmetic (see, for example, the discussion in [36]).

[28] If one reads the main content of the 13th Problem by replacing the word 'nomography' with 'algorithm', then the connection with the subject matter of this paper becomes fairly clear, [20], p.424:

> "[I]t is probable that the root of the equation of the seventh degree is a function of its coefficients which does not belong to the class of functions capable of nomographic construction, i.e., that it cannot be constructed by a finite number of insertions of functions of two arguments. In order to prove this, the proof would be necessary *that the equation of the seventh degree $f^7 + xf^3 + yf^2 + zf + 1 = 0$ is not solvable with the help of any continuous functions of only two arguments.* I may be allowed to add that I have satisfied myself by a rigorous process that there exist analytical functions of three arguments x, y, z which cannot be obtained by a finite chain of only two arguments."

Kolmogorov and Arnold refuted Hilbert's conjecture by constructing representations of continuous functions of several variables by the superposition of functions of one variable and sums of functions.

[29] In the opening lines of the section stating the 13th Problem, Hilbert gives an intuitive

terms of their *complexity* in a natural way: find those defining characteristics of a function, such that, the given function can be built up from simpler functions and simple operations. Hilbert's honed intuition suggested the formulation of the *13th Problem*; it was – like the *10th Problem* – solved 'negatively', by Kolmogorov and Arnold. The point to be emphasised here is, however, not the direct and obvious connection with computational complexity theory[30]. Essentially, Kolmogorov introduced the concept of '$\varepsilon-$entropy' of a metric space[31] 'to evaluate the order of increase of the volume of the [nomographic] table for an *increase in the accuracy* of [nomographic] tabulation.' In other words, in his work on approximation theory, preceding his work on algorithmic complexity theory by only a few years, Kolmogorov defined the 'size' of a finite body – actually a subset of a Banach space – in terms of its 'metric entropy'; on the other hand, in his work on algorithmic complexity theory, he defined the information content in a finite string in terms on 'entropy' (in the Shannon tradition), too.

Finally, the notion of randomness of finite strings, based on algorithmic complexity theory can, we now suggest, also be defined via the constructive notion of *lawless sequences (* or *Choice Sequences,* see [66]), first enunciated by Brouwer . In this way, we think the computable economist, trained in classical recursion theory or (in the inclusive sense), constructive analysis (say from [6]), supplemented by a mastery of one of the modern classics on algorithmic complexity theory ([29], [34] or [13]) and any standard classic on computational complexity theory (for eg., [39] or [47]), will be properly equipped to do justice to *Turing's visions*.

4 Computable Economics: Towards the Frontiers

> "*The theory of recursive functions properly belongs to number theory*; indeed, the theory of recursive functions is, so to speak, the function theory of number theory. ... The notion of recursive function marks off those functions whose values can be effectively calculated at every particular point; and *just those functions are useful in the natural sciences*. Though the variables of recursive functions do not run through all real numbers but only the natural

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idea of 'nomography', [20], p.424:

> "Nomography deals with the problem: to solve equations by means of drawings of families of curves depending on an arbitrary parameter."

Those of us who indulge in drawing *vector fields* might see the similarities!

[30] A beautiful discussion of approximation theory from this point of view – albeit implicitly – is given in an unfortunately little reference work by Vitushkin, [79]; a more technical and comprehensive survey of Kolmogorov's work on approximation theory is in [65].

[31] See [79], p. xiii and [30], chapters 9 & 10. 'Order of increase', 'increase in the accuracy', 'most favourable system of approximation', 'rapidity of convergence' are some of the phrases used in Kolmogorov approximation theory. These are the considerations that make approximation theoretical considerations naturally algorithmic and, therefore, also amenable to computational complexity analysis.

numbers, probability theory as well as quantum theory operates with functions of this latter kind; and recently recursive functions have begun to be applied in analysis too."

[**?**], p.7; italics added.

At least since Walras devised the *tâtonnement* process and Pareto's appeal to the market as a computing device, there have been sporadic attempts to find mechanisms to solve a system of supply-demand equilibrium equations, going beyond the simple counting of equations and variables. But none of these attempts to devise *mechanisms* to solve a system of equations were predicated upon the elementary fact that the data types – the actual numbers – realised in, and used by, economic processes were, at best, rational numbers. The natural equilibrium relation between supply and demand, respecting the elementary constraints of the equally natural data types of market – or any other kind of economy – should be framed as a Diophantine decision problems, and the way arithmetic games are formalised and shown to be effectively unsolvable in analogy with the *Unsolvability* of *Hilbert's Tenth Problem* (cf. [31]).

The Diophantine decision theoretic formalization is, thus, common to at least three kinds of computable economics: classical behavioural economics, algorithmic game theory in its incarnation as arithmetic game theory and elementary equilibrium economics. Even those, like Smale ([58]), who have perceptively discerned the way the problem of finding mechanisms to solve equations was subverted into formalizations of inequality relations which are then solved by appeal to (unnatural) non-constructive, uncomputable, fixed point theorems did not go far enough to realise that the data types of the variables and parameters entering the equations needed not only to be constrained to be non-negative, but also to be rational (or integer valued). Under these latter constraints, economics in its behavioural, game theoretic and microeconomic modes must come to terms with *absolutely (algorithmically) undecidable problems*. This is the cardinal message of the path towards computable economics.

Therefore, if orthodox algorithmic game theory, orthodox mechanism theory and computable general equilibrium theory have succeeded in computing their respective equilibria, then they would have to have done it with algorithms that are not subject to the strictures of the *Church-Turing Thesis* or do not work within the (constructive) proof-as-algorithm paradigm. This raises the mathematical meaning of the notion of algorithm in algorithmic game theory, orthodox mechanism theory and computable general equilibrium theory (and varieties of so-called computational economics). Either they are of the kind used in numerical analysis and so-called 'scientific computing' (as if computing in the recursion and constructive theoretic traditions are not 'scientific'; see [9] for a lucid definition and discussion of this seemingly innocuous concept) and, if so, their algorithmic foundations are, in turn, constrained by either the *Church-Turing Thesis* (as in [7]) or the (constructive) proof-as-algorithm paradigm; or, the economic system and its agents and institutions are computing the formally uncomputable and deciding the algorithmically undecidable (or are formal systems that are inconsistent or incomplete).

I believe Goodstein's algorithm, [17] could be the paradigmatic example for modelling rational - or integer - valued algorithmic (nonlinear) economic dynamics (see, for example, [40]). Every sense in which the notion of algorithm has been discussed above, for the path towards computable economics, is most elegantly satisfied by this line of research, a line that has by-passed the mathematical economics and nonlinear macrodynamics community. This is the only way I know to be able to introduce the algorithmic construction of an integer-valued dynamical system possessing a very simple global attractor, and with immensely long, effectively calculable, transients, whose existence is unprovable in *Peano Arithmetic*. Moreover, this kind of nonlinear dynamics, subject to *SSID*, ultra-long transients and possessing simple global attractors whose existence can be encapsulated within a classic *Gödelian, Diophantine*, decision theoretic framework, makes it also possible to discuss effective policy mechanisms (cf. [22]).

Kreisel's characteristically perceptive observation (see quote above, in the previous section), a plea for understanding the way to use the 'Goodstein algorithm' in economic dynamics and the economist's penchant for drawing curves and for working with numbers defined over the real numbers, convinces us that the most important frontier for computable economics is *computable analysis,* ([82]; coming down the [5] tradition) or *computable calculus* ([**?**], where a judicious combination of constructive logic and recursion theory is used). We have come to believe that every mathematically minded economist should be familiar with the *graph theorem of classical recursion theory* ([36], p. 135-6), and not simply be bamboozled by the Dirichlet-Kuratowski graph concept. The interaction between *recursive* and *recursively enumerable* sets, computable functions and *functions 'plottable' on a digital computer's screen* should be made clear to all students of economics, almost more importantly than teaching them probability theory, statistics and the like. This is implicit in some of the claims about the notion and definition of computation universality we have routinely been using in classical computable economics.

With an integration of *classical recursion theory, computable analysis* and a familiarity with the framework of *Diophantine Decision Problems*, and the suggestions in the previous section on mastering the double duality between algorithmic complexity and computational complexity, on the one hand, and between classical recursion theory and constructive analysis, on the other, *classical* computable economics will be ready to embark on the path towards *modern* computable economics, where not only the *theory* of the computer will be an underpinning of economic theory; but also the empirical use of the hardware, the pixels and the resolution that make the screen as much a part of the computable economist's 'box of tools' as its theory, will enrich the experiences of being educated to be a computable economist.

It is incumbent upon us to make the attempt to prepare for a 'computable and constructive' future, by writing the 'sensible textbooks' ([57]), for the next – or future – generations of students, who will be the harbingers of the computable approach to economics. It is only this way we can pay *homage* to **Turing's genius**.

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