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COMPUTABILITY THEORY IN ECONOMICS FRONTIERS AND A RETROSPECTIVE[♣]

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[♣]I would like to dedicate this paper to Michael Rabin for his 80th birthday year, albeit more than one year too late, as a humble homage to the person I consider the founding father of *Computable Economics*. His effectivization of the Gale-Stewart Game remains the model methodological contribution to the field for which I coined the name *Computable Economics* more than 20 years ago. His classic of computable economics stands in the long and distinguished tradition that goes back to classics by Zermelo, Banach & Mazur, Steinhaus and Euwe. A part of this heritage will be discussed in the main body of the paper. I should add that I have never met Michael Rabin and do not know him personally - or professionally - at all. I know him only through his remarkable contributions to computability and computational complexity theories.

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Abstract

This is an outline of the origins and development of the way computability theory was incorporated into formal economic theory. I try to place in the context of the development of computable economics, some of the classics of the subject as well as those that have, from time to time, been credited with having contributed to the advancement of the field. Speculative methodological thoughts and reflections suggest directions in which fruitful research could proceed to reduce the current deficit in the epistemology of computation in economics. Finally, thoughts on where the frontiers of computable economics are, and how to move towards them, conclude the paper. In a precise sense – both historically and analytically – it would not be an exaggeration to claim that both the origins of computable economics and its frontiers are defined by two classics, both by Banach and Mazur: that one page masterpiece by Banach and Mazur ([6]) and the unpublished Mazur conjecture of 1928, and its unpublished proof by Banach ([48], ch. 6 & [78], ch. 1, §.6). For the undisputed original classic of computable economics is Rabin's *effectivization* of the Gale-Stewart game ([51]; [22]); the frontiers, as I see them, are defined by recursive analysis and constructive mathematics, underpinning computability over the computable *and constructive* reals and providing computable foundations for the economist's Marshallian penchant for *curve-sketching* ([10]; [**?**]; [27]; [94] and, in general, the contents of **Theoretical Computer Science**, Vol. 219, Issue 1-2). The former work has its roots in the Banach-Mazur game (cf. [48], especially p.30), at least in one reading of it; the latter in ([6]), as well as other, earlier, contributions, not least by Brouwer.

Key Words: Computability, Effectivization, Constructivity, Uncomputability, Computable Economics

1 A Setting for Computable Economics

> "The next step in analysis[1], I would conjecture, is a more consistent assumption of *computability* in the formulation of economic hypotheses. This is likely to have its own difficulties because, of course, not everything is computable, and there will be in this sense an inherently unpredictable element in rational behavior."
>
> [2], p.S398; italics added.

This 'next step in [economic] analysis', conjectured by the *doyen* of mathematical economics, Kenneth Arrow, has *not* been taken by economic theorists or, more pertinently, by anyone claiming to be a *computational* economist, *computable* general equilibrium theorist, applied *computable* general equilibrium theorist, *algorithmic* game theorist, so-called agent-based economic and financial modeller or any variety of *DSGE*[2] theorist. Indeed, not too long after the famous, and decidedly *non-computable* and *non-constructive*, *Arrow-Debreu classic* was published ([3], the trio of outstanding mathematical economists, Arrow, Karlin and Scarf, cautioned economists against facile conflation of *existence theorems* and *effectively computable solutions*:

> "The term 'computing methods' is, of course, to be interpreted broadly as the mathematical specification of algorithms for arriving at a solution (optimal or descriptive), rather than in terms of precise programming for specific machines. Nevertheless, we want to stress that solutions which are not *effectively computable* are not properly solutions at all. *Existence theorems and equations* which must be satisfied by optimal solutions are useful tools toward arriving at *effective solutions*, but *the two must not be confused*. Even iterative methods which lead in principle to a solution cannot be regarded as acceptable if they involve computations beyond the possibilities of present-day computing machines."
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> [4] p.17; italics added.

Despite this early 'warning' by three of the pioneering mathematical economists of economic theorising in the non-computable mode, only in the sadly aborted research program on *effectively constructive economics* by Alain Lewis and in my *computable economics*, have there been *systematic* and coherent attempts to take Arrow's conjecture seriously[3]. As far as I am concerned, Simon

[1] Clearly, the context implies that this refers to 'the next step in *economic* analysis'.

[2] **D**ynamic **S**tochastic **G**eneral **E**quilibrium.

[3] *Computable Economics* is a name I coined in the early 1980s, from the outset with the intention of encapsulating computability and constructivity assumption in economic theory. My earliest recollection is 1983, when I announced a series of graduate lectures on *Turing and his Machine for Economists*, in the department of economics at the European University Institute. Only one person signed up for the course, Henrietta Grant-Peterkin, one of our valued departmental secretaries! The course was still-born.

([62]), together with Michael Rabin ([51][4]) and Alain Lewis, are the undisputed pioneers of *Computable Economics*, and both of these classics appeared in the public domain before ([4]). In [82] it was pointed out that (pp. 25-6):

> "[Simon's] path towards a broader base for economics .... stressed two empirical facts (quotes are from [64], p. x):
>
> (I). 'There exists a basic repertory of mechanisms and processes that Thinking Man uses in all the domain in which he exhibits intelligent behavior.';
>
> (II). 'The models we build initially for the several domains must all be assembled from this same basic repertory, and *common principles of architecture* must be followed throughout.' (italics added);"
>
> It is at this point that I feel Simon's research program pointed the way toward computable economics in a precise sense. .....
>
> Instead, the direction Simon took codified his research program in terms of the familiar notions of bounded rationality and satisficing [underpinned by computational complexity theory] ..
>
> I remain convinced that, had Simon made the explicit recursion-theoretic link at some point in the development of his research program, computable economics would have been codified much earlier."

After reading [82], Simon wrote me as follows (italics added):

> " As the book makes clear, my own journey through bounded rationality *has taken a somewhat different path.* Let me put it this way. There are many levels of *complexity in problems*, and corresponding boundaries between them. *Turing computability is an outer boundary*, and as you show, any theory that requires more power than that surely is irrelevant to any useful definition of human rationality. ....
>
> Finally, we get to the empirical boundary, measured by laboratory experiments on humans and by observation, of the level of complexity that humans actually can handle, with and without their computers, and - perhaps more important – what they actually do to solve problems that lie beyond this strict boundary even though they are within some of the broader limits.
>
> The latter is an important point for economics, because we humans spend most of our lives making decisions that are far beyond any of the levels of complexity we can handle exactly; and this is where satisficing, floating aspiration levels, recognition and heuristic search, and similar devices for arriving at good-enough decisions take over. A parsimonious economic theory, and an empirically verifiable one, shows how human beings, using very simple procedures,

[4] In one of the most elegantly written 'eternal' classics of recursion theory, Hartley Rogers ([53]), the one blemish I found is the relegation of Rabin's results to a minor problem (p.121, ex. 8.5), with the unfortunate comment: 'This is a special and trivial instance of a general theorem about games'!

reach decisions that lie far beyond their capacity for finding exact solutions by the usual maximizing criteria "

Simon chose to work within the '*empirical* boundary', recognising immediately that computable economics was an attempt at defining, effectively, the relevance of the 'outer boundary' for formalisation in economic theory.

Alain Lewis is a contemporary pioneer, whose research program on *Effectively Constructive Mathematics* ([36], [35]) had an immense flowering in the years between the mid-1980s and the early 1990s[5]. In his remarkably prescient *Monograph* (manuscript), [35], the elegant interpretation of (what he calls) *Rabin's Theorem* ([51]), brings together the three undisputed pioneers of computable economics, i.e., *Herbert Simon*, *Michael Rabin* and *Alain Lewis* himself, in one fell swoop, so to speak (*ibid*, p. 84; underlining in the original):

> "M.O. Rabin .... was the first ... to make a significant application of recursion theory to the theory of games. In Rabin ([51]) it is remarked that 'It is obvious that not all games that are considered within the theory of games are actually playable by human beings.'[6] Here we find H. Simon's [[62]] concept of <u>bounded rationality</u> as a hidden theme, for the point of Rabin's inquiry is to determine if certain games of the Gale-Stewart variety can be won consistently by Turing Machines that serve as surrogate players. To quote Rabin [[51], p. 147] once more: 'The question arises as to what extent the existence of winning strategies makes a win-lose [i.e., zero-sum] game trivial to play. Is it always possible to build a computer which will play the game and consistently win?'
>
> What Rabin is doing here is to provide an interpretation of Simon's concept of bounded rationality that is <u>computational in character</u>. The significance of [[51]] is that the techniques of recursion theory are used to fix a precise interpretation of computability within Church's Thesis."

Although the last sentence in the above important observation by Lewis is, perhaps, less than felicitous, the true significance of Lewis's insight was to realise

[5] That this blossoming withered away quite abruptly, around 1993, remains a mystery to many of us. Even as late as 1992, I received a warm, very personal, letter from Lewis (May 21, 1992), which ended with the query (referring to Turing Machine interpretations, applications and formalisations in economic theory):

> "Why are people so afraid to do what von Neumann actually had in mind?"

In personal conversations with me, in early 1992, Lewis was scathingly bitter about his experiences with submissions to the **Econometrica**, in particular, whose editorial policy, *according to him*, was to discourage any submission that had anything to do with applying Gödelian or Turing Machine results in economics.

[6] The exact quotation is ([51], p. 147):

> "It is quite obvious that not all games which are considered in the theory of games can actually be played by human beings."

that Simon's concept of bounded rationality had to be given computational content; that Lewis did not also realise that Simon did give it this content from the outset is besides the point. But to give the notion of bounded rationality computational content in the context of games played by computing machines is one thing; to interpret bounded rationality as encapsulated in *finite automata* is quite another thing. Fortunately, Lewis did not fall into the latter trap, one which many distinguished game theorists almost willingly embraced ([47]).

However, long before Lewis recognised, perceptively, Rabin's 'hidden theme' in Simon's concept of 'bounded rationality', another classic, little known in mathematical economic or game theoretic circles, linked *effective computability* with *bounded rationality* in the context of a remarkably original contribution to a political science basis for *wei-ch'i* – better known in the 'west' as **GO**. Scott Boorman ([9], p. 210, footnote 6; italics added) observed:

> "In theory, of course, the counterstrategy [available to an opponent] is *effectively computable.*"

Having read this only a couple of years ago, I wrote Professor Boorman, on 25 June, 2009, wondering whether he had 'a formal result to this effect or, if not, whether [he] could direct me to any other work that derives this result formally.' I added that although I could 'believe ..it is possible to prove, formally and non-constructively, there exists such a "counterstrategy", I very much [doubted] that 'it is provable that such a strategy is *effectively computable.*'

Professor Boorman's gracious and fairly immediate response was most illuminating[7]:

> "Regarding the specific matter you raise, I've taken a look at note 6 on p. 210 of my 1969 book you point to, and have the following thoughts (presented here in a spirit that builds in part on your important statement that "it is never useless to know what the pioneer did – and why he did it"). [I]t's clear to me – both from my memory & from context – that my main focus in note 6 was on *cognitive, not mathematical, limits of computation.* Although Herbert Simon isn't mentioned by name in note 6 (he appears earlier, on p. 187), in essence I'm talking about *a class of bounded rationality issues.*"

It was only a few months after I received the above response from Professor Boorman that I was able to interpret Simon's notion of bounded rationality, in conjunction with satisficing, in the framework of (*metamathematical*) *decision problems (* see section 2.1, below). This made it possible for me to understand and formally demonstrate the effectively computable content of boundedly rational agents implementing satisficing behaviour.

The point missed by Lewis in his handsome tribute to Rabin is that this classic came down in the great tradition of *alternating games* (see [81]), begun

[7]E-mail from Boorman to Velupillai, dated 7 July, 2009; italics added.

by Zermelo at the beginning in ([96]), on the one hand; and, on the other hand, down the even nobler and more ancient tradition of what is now called *combinatorial games* (see the recent elegant, and eminently readable, [44] for a fine exposition of the history and origins of this field, with copious references). But there are many eminent game theorists who feel able to claim Zermelo as a precursor of orthodox game theory. In some senses – particularly with regard to von Neumann's original min-max result and to the sustained non-constructive and uncomputable methodology that underpins formal, orthodox, game theory - this claim many have a modicum of truth to it.

My own 'take' on Rabin's classic as the fountainhead of computable economics is its pedagogic value in providing a tutorial on how to effectivise a non-effective framework in orthodox theory – whether economic or game theoretic. This is what I have emphasised in [81]. But, of course, it has also led to a revitalisation of both a part of recursion theory (see p. 254 in the excellent – although slightly dated – survey by Telgárski, [**?**], of recursion theoretic work inspired by Banach-Mazur games for some of the early and classic references), and a reflection on the possibility of avoiding reliance on the *axiom of choice*[8] (see below, the comment on the *axiom of determinacy*).

The von Neumann paper of 1928 ([91]), the 'official' fountainhead for *orthodox game theory*, introduced, and etched indelibly, to an unsuspecting and essentially non-existent Mathematical Economics community, what has eventually come to be called '*Hilbert's Dogma*'[9], 'consistency $\Leftrightarrow$ existence'. This became – and largely remains – the mathematical economist's *credo*. Hence, too, the inevitable schizophrenia of 'proving' existence of equilibria, first, and looking for methods to *construct* and *compute* them at a second, entirely unconnected, stage. Thus, too, the indiscriminate appeals to the *tertium non datur* – and its implications – in 'existence proofs', on the one hand, and the ignorance about the nature and foundations of constructive mathematics and computability theory, on the other.

But it was not as if von Neumann was *not* aware of Brouwer's opposition to 'Hilbert's Dogma', even as early as 1928, although there is reason to suspect that something peculiar may have been going on. Hugo Steinhaus observed, with perplexity, ([72]):

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[8] Even distinguished mathematical economists seem to think, routinely, that a reliance on the axiom of choice is unavoidable in rational and social choice theory (eg., [73]) or mathematical economics, particularly in the uncritical appeal to the Hahn-Banach theorem (eg., [14]). It is puzzling that very few, or none, in the mathematical economics community seem to be aware that perfectly respectable constructive versions of the Hahn-Banach Theorem can be invoked – but, of course, the price one has to pay is that the fundamentals of economic theory have to be reformalised constructively, too. This is not impossible; just inconvenient!

[9] In van Dalen's measured, scholarly, opinion, [80], pp. 576-7 (italics added):

> "Since Hilbert's yardstick was calibrated by the continuum hypothesis, *Hilbert's dogma, 'consistency $\Leftrightarrow$ existence'*, and the like, he was by definition right. *But if one is willing to allow other yardsticks*, no less significant, but based on alternative principles, then Brouwer's work could not be written off as obsolete nineteenth century stuff."

> "[My] inability [to prove the minimax theorem] was a consequence of the ignorance of Zermelo's paper in spite of its having been published in 1913. .... J von Neumann was aware of the importance of the minimax principle [in [91]]; it is, however, *difficult to understand the absence of a quotation of Zermelo's lecture in his publications.*"
>
> ibid, p. 460; italics added

Why didn't von Neumann refer, in 1928, to the Zermelo-tradition of (*alternating*) *games*? van Dalen, in his comprehensive, eminently readable and scrupulously fair biography of Brouwer, [80], p. 636, noted (italics added), without additional comment that:

> "In 1929 there was another publication in the intuitionistic tradition: an intuitionistic analysis of the game of chess by Max Euwe[10]. It was a paper in which the game was viewed as a spread (i.e., a tree with the various positions as nodes). Euwe carried out *precise constructive estimates* of various classes of games, and considered the influence of the rules for draws. When he wrote his paper he was not aware of the earlier literature of Zermelo and Dénès König. *Von Neumann called his attention to these papers, and in a letter to Brouwer, von Neumann sketched a classical approach to the mathematics of chess, pointing out that it could easily be constructivized.*"

Why didn't von Neumann provide this 'easily constructivized' approach – then, or later? Perhaps it was easier to derive propositions appealing to the *tertium non datur,* and to 'Hilbert's Dogma', than to do the hard work of *constructing estimates* of an algorithmic solution*, as* Euwe did[11]? Perhaps it was easier to continue using the *axiom of choice* than to construct new axioms – say the *axiom of determinacy*[12] – as Steinhaus and Mycielski did ([38])? Whatever the reason, the fact remains that the von Neumann legacy was indisputably a

[10]In a strange lapse, van Dalen refers to Euwe, 1929, without giving the exact details of the reference in his excellent bibliography. The exact reference is [21]. Max Euwe was the fifth World Chess Champion, between 1935-1937, having defeated Alexander Alekhine, on December 15, 1935. A translation of this comprehensively neglected classic, with comments, and placing it in the context of the development of Arithmetical Games and orthodox game theory, is forthcoming in [?].

[11]At the end of his paper Euwe reports that von Neumann brought to his attention the works by Zermelo and König, after he had completed his own work (ibid, p. 641). This further substantiates the perplexity reported by Steinhaus (above) on the absence of any reference to Zermelo in von Neumann's official publications of the time. In any case, Euwe then goes on (italics added):

> "*Der gegebene Beweis is aber nicht konstruktive*, d.h. es wird keine Methode angezeigt, mit Hilfe deren der gewinnweg, wenn überhaupt möglich, *in endlicher Zeit konstruiert werden kann.*"

Perhaps Michael Rabin was aware of this remarkable paper by Max Euwe, which may be why he imposed the kind of finiteness condition to which the latter refers in his last phrase!

[12]For the aims of this particular expository essay, the introduction of this axiom is particularly relevant. The point I wish to make is best described in Gaisi Takeuti's important

legitimization of 'Hilbert's Dogma' and the indiscriminate use of the axiom of choice in mathematical economics and game theory.

I began to think of Game Theory in algorithmic modes – but not what is today referred to as *Algorithmic Game Theory* – after realizing the futility of algorithmising the uncompromisingly subjective von Neumann-Nash approach to game theory and beginning to understand the importance of *Harrop's theorem* ([26], see also [76], where the indeterminacy of even finite games, using Harrop's Theorem, is outlined). This realization came after an understanding of *effective playability* in arithmetical games, developed elegantly by Michael Rabin.

The brief, rich and primarily recursion theoretic framework of Harrop's classic paper requires a deep understanding of the rich interplay between *recursivity and constructive representations* of **finite sets** that are *recursively enumerable.* There is also an obvious and formal connection between the notion of a finite combinatorial object, whose complexity is formally defined by the uncomputable Kolmogorov measure of complexity, and the results in Harrop's equally pioneering attempt to characterise the recursivity of finite sets and the resulting indeterminacy – undecidability – of a Nash equilibrium *even in the finite case.* To the best of my knowledge this interplay has never been mentioned or analysed in the mathematical economic or game theoretic literature.

When I conceived the notion of computable economics in the early 1980s, I had in mind both constructive and computable mathematics as bases for the formalization of economic theory, which is entirely consistent with Arrow's above conjecture. I was blissfully ignorant of the pioneering works by Rabin and Lewis, till about the late 1980s. Also, the important work by Douglas Bridges based on constructive mathematics were unknown to me when I was fashioning computable economics including constructive assumptions and interpretations. It is a pleasure, now, to acknowledge his absolute priority in constructive economics.

Finally, anyone even remotely familiar with Conway's characteristically clear note on *A Gamut of Game Theories* ([16]) and Turing's classic on *Solvable and Unsolvable Problems* ([77]), and Herbert Simon's kind of behavioural economics – called classical behavioural economics in this paper – will know that there is an almost formal duality between problem solving and (combinatorial) games. This is not a theme space allows me to develop, but it needs to be pointed out that any future for computable economics will have to enlarge on this aspect of the interaction between recursion theory, combinatorial games, Ramsey theory and behavioural economics.

The paper is organised as follows. The next section is a retrospective of

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observation ([74], pp. 73-4; italics added):

> "There has been an idea, which was originally claimed by Gödel and others, that, if one added an axiom which is a strengthened version of the existence of a measurable cardinal to existing axiomatic set theory, then various mathematical problems might all be resolved. Theoretically, nobody would oppose such an idea, but, in reality, *most set theorists felt it was a fairy tale and it would never really happen.* But it has been *realized by virtue of the axiom of determinateness*, which showed Gödel's idea valid."

some of the results obtained under the rubric of computable economics. The whole section is sub-divided into six sub-sections, each dealing with some of the classic results in computable economics. The first four sub-sections contain so-called 'negative' results – uncomputability, nonconstructivity and undecidability of classic mathematical economic results. The last two sections contain more positive results. The third sections outlines my own view of the frontiers of computable economics. The main vision here is the hope that 'the next step in computable economic analysis would be a more consistent' consideration of recursive or computable analysis, particularly in macroeconomic dynamics.

2 Computable Economics: A Retrospective[13]

> "[The] adoption of the *infinitary, nonconstructive*, *set theoretic*, algebraic, and structural methods that are characteristic to modern mathematics [. . . .] were controversial, however. At issue was not just whether they are consistent, but, more pointedly, whether they are meaningful and appropriate to mathematics. After all, *if one views mathematics as an essentially computational science*, then arguments without computational content, whatever their heuristic value, are not properly mathematical. .. [At] the bare minimum, we wish to know that the universal assertions we derive in the system will not be contradicted by our experiences, and *the existential predictions will be borne out by calculation.* This is exactly what *Hilbert's program*[14] was designed to do."
>
> [5], pp. 64-5; italics added

Thus, my claim is that the *existential predictions* made by the purely theoretical part of mathematical economics, game theory and economic theory '*will* [*not*] *be borne out by calculations.'* There is, therefore, a serious *epistemological deficit* – in the sense of economically relevant knowledge that can be processed and accessed computationally and experimentally – in all of the above approaches, claims to the contrary notwithstanding, that is unrectifiable without *wholly abandoning their current mathematical foundations.* This is an *epistemological deficit* even before considering the interaction between appeals to infinite – even uncountably infinite – methods and processes in proofs, where both the universal and existential quantifiers are freely used in such contexts, and the *finite* numerical instances[15] with which they are, ostensibly, 'justified'.

[13] A part of these results appeared in this *Journal* in [?] and [84]. They are summarised here, in a concise way, just to provide a context for a coherent setting for the narrative.

[14] I have tried to make the case for interpreting the philosophy and methodology of mathematical economics and economic theory in terms of the discipline of *Hilbert's program* in [?].

[15] Serényi's ([59]) very recent reflections and results on this issue will play an important part in the theoretical underpinnings to be developed in this project (p.49; italics added):

> "An argument deriving the truth of a universal arithmetical sentence from that of its numerical instances suggests that the truth of the numerical instances has

This epistemological deficit requires even 'deeper' mathematical and philosophical considerations in *Cantor's Paradise*[16] of ordinals[17] , where combinatorics, too, have to be added to computable and constructive worlds to make sense of claims by various mathematical economists and agent based modeling practitioners.

Against this backdrop, within the framework of what I will now call classical computable economics, the following are some of the results that have been derived[18]:

1. Nash equilibria of (*even*) *finite* games are constructively indeterminate.
2. The Arrow-Debreu equilibrium is uncomputable (and its existence is proved nonconstructively).
3. The Uzawa Equivalence Theorem is uncomputable and nonconstructive.
4. Computable General Equilibria are neither computable nor constructive.
5. The Two Fundamental Theorems of Welfare Economics are Uncomputable and Nonconstructive, respectively.
6. The Negishi method is proved nonconstructively and the implied procedure in the method is uncomputable.
7. There is no effective procedure to generate preference orderings.
8. Rational expectations equilibria are uncomputable and are generated by uncomputable and nonconstructive processes.
9. Policy rules in macroeconomic models are noneffective.

> some kind of *epistemological priority* over the truth of the sentence itself: our knowledge of the truth of the sentence stems from the fact that we know all its numerical instances to be true. .. I shall show that it is just the other way around. ... *[T]he source of our knowledge of the truth of the totality of its numerical instances is the truth of the sentence itself.*"

[16]Hilbert did not want to be driven out of '*Cantor's Paradise*' ([28]; p.191):

> 'No one shall drive us out of the paradise which Cantor has created for us.'

To which the brilliant 'Brouwerian' response, if I may be forgiven for stating it this way, by Wittgenstein was ([93]; p.103):

> 'I would say, "I wouldn't dream of trying to drive anyone out of this paradise." I would try to do something quite different: I would try to show you that it is not a paradise – so that you'll leave of your own accord. I would say, You're welcome to this; just look about you." '

[17]Where '*Ramsey Theory*', '*Goodstein Sequences*' and the '*Goodstein theorem*', reign supreme. In work in progress these issues are dealt with in some detail, as they pertain to bridging the '*epistemological deficit*' in economic theoretical discourse in the mathematical mode.

[18]Apart from the twelfth result, which, as emphasised in the opening section, is due to the pioneering work of Michael Rabin ([51]) in 1957, the rest are due to this author. The first was suggested by Francisco Doria in some of his philosophical writings. See also [90].

10. Recursive Competitive Equilibria (RCE), underpinning the Real Business Cycle (RBC) model and, hence, the *Dynamic Stochastic General Equilibrium* (DSGE) benchmark model of Macroeconomics, are uncomputable.

11. Dynamical systems underpinning growth theories are incapable of computation universality.

12. There are games in which the player who in theory can always win cannot do so in practice because it is impossible to supply him with effective instructions regarding how he/she should play in order to win.

13. The theoretical benchmarks of Algorithmic Game Theory are uncomputable and non-constructive.

14. Boundedly rational agents,satisfying, formalised within the framework of (metamathematical) decision problems are capable of effective procedures of rational choice.

In the next subsection I outline the computability theoretic background against which # 14 can be demonstrated. In subsections 2, 3, 4 and 5, results # 3 $\sim$ # 5 are outlined, with brief formal demonstrations. The final subsection is a brief outline of classical computable economics, in retrospective mode.

2.1 Classical Behavioural Economics - Computable Foundations

"If we hurry , we can catch up to Turing on the path he pointed out to us so many years ago."

Herbert Simon, [66], p. 101.

I coined the phrase *classical behavioural economics* to characterise the kind of behavioural economics pioneered by Herbert Simon, which was underpinned, always and at any and every level of theoretical and applied analysis, by a model of computation. Invariably, although not always explicitly, it was *Turing's model of computation*. Three of the undisputed frontier researches in 'modern' behavioural economics, Colin Camerer, George Lowenstein and Matthew Rabin ([12]), in their *Preface* to **Advances in Behavioral Economics** state:

> "Twenty years ago [i.e., 1984], behavioural economics did not exist as a field...... Richard Thaler's 1980 article 'Toward a Theory of Consumer Choice', of the remarkably open-minded (for its time) Journal of Economic Behavior and Organization, is considered by many to be the first genuine article in *modern behavioural economics*."

To highlight the difference between modern behavioural economics, which is never underpinned by a model of computation, and the kind of behavioural economics that was pioneered and practiced by Simon and his associates and followers, I decided to refer to the latter as practitioners of classical behavioural economics.

The fundamental focus in classical behavioural economics is on decision problems faced by human problem solvers, the latter viewed as information processing systems. All of these terms are given computational content, *ab initio.* But given the scope of this paper I shall not have the possibility of a full characterisation. The ensuing 'bird's eye' view must suffice for now[19].

A *decision problem* asks whether there exists an *algorithm* to *decide* whether a mathematical assertion does or does not have a proof; or a formal problem does or does not have an algorithmic solution. Thus the characterization makes clear the crucial role of an underpinning *model of computation*; secondly, the answer is in the form of a *yes/no* response. Of course, there is the third alternative of '*undecidable*', too. It is in this sense of *decision problems* that we interpret the word 'decisions' here.

As for 'problem solving', I shall assume that this is to be interpreted in the sense in which it is defined and used in the monumental classic by Newell and Simon ([43]).

Finally, the *model of computation* is the *Turing model*, subject to the *Church-Turing Thesis.*

To give a rigorous mathematical foundation for bounded rationality and satisficing, as decision problems[20], it is necessary to underpin them in a dynamic model of choice in a computable framework. However, these are not two separate problems. Any formalization underpinned by a model of computation in the sense of computability theory is, dually, intrinsically dynamic.

Remark 1 *Decidable-Undecidable, Solvable-Unsolvable, Computable-Uncomputable, etc., are concepts that are given content algorithmically.*

Now consider the Boolean formula:

[19] Some details are discussed in greater and more rigorous depth in [85]

[20] The three most important classes of decision problems that almost characterise the subject of computational complexity theory, underpinned by a model of computation – in general, the model of computation in this context is the *Nondeterministic Turing Machine* – are the **P**, **NP** and **NP-Complete** classes. Concisely, but not quite precisely, they can be described as follows:

1. **P** defines the class of computable problems that are solvable in time bounded by a *polynomial function of the size of the input*;
2. **NP** is the class of computable problems for which a *solution* can be *verified in polynomial time*;
3. A computable problem lies in the class called **NP-Complete** if every problem that is in **NP** can be *reduced to it in polynomial time.*

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \{\neg x_2\}) \wedge (x_2 \vee \{\neg x_3\}) \wedge (x_3 \vee \{\neg x_1\}) \wedge (\{\neg x_1 \vee \{\neg x_2\} \vee \{\neg x_3\}) \tag{1}$$

Remark 2 *Each subformula within parenthesis is called a clause; The variables and their negations that constitute clauses are called literals; It is 'easy' to 'see' that for the truth value of the above Boolean formula to be $t(x_i) = 1$, all the subformulas within each of the parenthesis will have to be true. It is equally 'easy' to see that no truth assignments whatsoever can satisfy the formula such that its global value is true. This Boolean formula is unsatisfiable.*

Problem 3 *SAT – The Satisfiability Problem*

Given m clauses, $C_i(i = 1, \ldots, m)$, containing the literals (of) $x_j(j = 1, \ldots, n)$, *determine* if the formula $C_1 \wedge C_2 \wedge \ldots\ldots\ldots \wedge C_m$ is *satisfiable*.

Determine means 'find an (efficient) algorithm'. To date it is not known whether there is an *efficient* algorithm to solve *the satisfiability problem* – i.e., to determine the truth value of a Boolean formula. In other words, it is not known whether $SAT \in \mathbf{P}$. But:

Theorem 4 $SAT \in \mathbf{NP}$

Definition 5 *A Boolean formula consisting of many clauses connected by conjunction (i.e., $\wedge$) is said to be in Conjunctive Normal Form (CNF).*

Finally, we have Cook's famous theorem:

Theorem 6 *Cook's Theorem*
SAT is **NP** *– Complete*

It is in the above kind of context and framework within which we are interpreting Simon's vision of behavioural economics. In this framework optimization is a very special case of the more general decision problem approach. The real mathematical content of *satisficing*[21] is best interpreted in terms of the satisfiability problem of computational complexity theory, the framework used by Simon consistently and persistently - and a framework to which he himself made pioneering contributions.

Finally, there is the computably underpinned definition of bounded rationality.

[21] In [67], p. 295, Simon clarified the *semantic* sense of the word *satisfice*:

> "The term 'satisfice', which appears in the *Oxford English Dictionary* as a Northumbrian synonym for 'satisfy', was borrowed for this new use by H. A. Simon (1956) in 'Rational Choice and the Structure of the Environment' [i.e, [63]]".

Theorem 7 *The process of rational choice by an economic agent is formally equivalent to the computing activity of a suitably programmed (Universal) Turing machine.*

Proof. By construction. See §3.2, pp. 29-36, *Computable Economics* [[82]] ■

Remark 8 *The important caveat is 'process' of rational choice, which Simon – more than anyone else – tirelessly emphasized by characterizing the difference between 'procedural' and 'substantive' rationality; the latter being the defining basis for Olympian rationality ([65], p.19), the former that of the computationally underpinned problem solver facing decision problems. Any decision – rational or not – has a time dimension and, hence, a content in terms of some process. In the Olympian model the 'process' aspect is submerged and dominated by the static optimization operator, By transforming the agent into a problem solver, constrained by computational formalisms to determine a decision problem, Simon was able to extract the procedural content in any rational choice. The above result is a summary of such an approach.*

Definition 9 *Computation Universality of a Dynamical System*

A dynamical system – discrete or continuous – is said to be capable of computation universality if, using its initial conditions, it can be programmed to simulate the activities of any arbitrary Turing Machine, in particular, the activities of a Universal Turing Machine.

Lemma 10 *Dynamical Systems capable of Computation Universality can be constructed from Turing Machines*

Proof. See [82] ■

Theorem 11 *Non-Maximum Rational Choice*

No trajectory of a dynamical system capable of universal computation can, in any 'useful sense' (see Samuelson's Nobel Prize lecture, [55]), be related to optimization in the Olympian model of rationality.
Proof. See [82] ■

Theorem 12 *Boundedly rational choice by an information processing agent within the framework of a decision problem is capable of computation universality.*

Proof. An immediate consequence of the definitions and theorems of this subsection. ■

Remark 13 *From this result, in particular, it is clear that the Boundedly Rational Agent, satisficing in the context of a decision problem, encapsulates the only notion of rationality that can 'in any useful sense' be defined procedurally.*

I have only scratched a tiny part of the surface of the vast canvass on which Simon sketched his vision of a computably underpinned behavioural economics. Nothing in Simon's behavioural economics – i.e., in *Classical Behavioural Economics* – was devoid of computable content. There was – is – never any epistemological deficit in any computational sense in classical behavioural economics.

2.2 *The Nonconstructive Aspects of Brouwer's and Kakutani's Fix Point Theorems*

In Scarf's classic book of 1973 there is the following characteristically careful caveat to any unqualified claims to *constructivity* of the algorithm he had devised:

> "In applying the algorithm it is, in general, *impossible* to select an ever finer sequence of grids and a convergent sequence of subsimplices. An algorithm for a digital computer must be basically finite and cannot involve an infinite sequence of successive refinements. .... *The passage to the limit is the nonconstructive aspect of Brouwer's theorem*, and we have no assurance that the subsimplex determined by a fine grid of vectors on S contains or is even close to a true fixed point of the mapping."
>
> [56], p.52; italics added

An algorithm, by definition, is a finite object, consisting of a finite sequence of instructions. However, such a finite object is perfectly compatible with 'an infinite sequence of successive refinements' ([56], p. 52), provided a stopping rule associated with a clearly specified and verifiable approximation value is part of the sequence of instructions that characterize the algorithm. Moreover, it is *not* 'the passage to the limit [that] is the nonconstructive aspect of Brouwer's [fix point] theorem' (ibid, p.52)[22]. Instead, the sources of non-constructivity are the undecidable disjunctions - i.e., appeal to the *law of the excluded middle* in infinitary instances - intrinsic to the choice of a convergent subsequence in the

[22] In [57], p. 1024, Scarf is more precise about the reasons for the failure of constructivity in the proof of Brouwer's fix point theorem:

> "In order to demonstrate Brouwer's theorem completely we must consider a sequence of subdivisions whose mesh tends to zero. Each such subdivision will yield a completely labeled simplex and, *as a consequence of the compactness* of the unit simplex, there is a convergent subsequence of completely labeled simplices all of whose vertices tend to a single point x^*. (This is, of course, the non-constructive step in demonstrating Brouwer's theorem, rather than providing an approximate fixed point)."

There are two points to be noted: first of all, even here Scarf does not pinpoint quite precisely to the main culprit for the cause of the non-constructivity in the proof of Brouwer's theorem; secondly, nothing in the construction of the algorithm provides a justification to call the value generated by it to be an approximation to x^*. In fact the value determined by Scarf's algorithm has no theoretically meaningful connection with x^* (i.e., to p^*) for it to be referred to as an approximate equilibrium.

use of the Bolzano-Weierstrass theorem[23] and an appeal to the *law of double negation* in an infinitary instance during a *retraction*. The latter reliance invalidates the proof in the eyes of the Brouwerian constructivists; the former makes it constructively invalid from the point of view of every school of constructivism, whether they accept or deny intuitionistic logic.

Brouwer's proof of his celebrated fix point theorem was indirect in two ways: he proved, first, the following:

Theorem 14 *Given a continuous map of the disk onto itself with no fixed points, $\exists$ a continuous retraction of the disk to its boundary.*

Having proved this, he then took its *contrapositive:*

Theorem 15 *If there is no continuous retraction of the disk to its boundary then there is no continuous map of the disk to itself without a fixed point.*

Using the logical principle of equivalence between a proposition and its contrapositive (i.e., logical equivalence between theorems 7 & 8) and the law of double negation ($\nexists$ a continuous map with **no** fixed point = $\exists$ a continuous map with a fixed point) Brouwer demonstrated the existence of a fixed point for a continuous map of the disk to itself. This latter principle is what makes the proof of the Brouwer fix point theorem via retractions (or the non-retraction theorem) essentially unconstructifiable. Scarf's attempt to discuss the 'relationship between these two theorems [i.e., between the non-retraction and Brouwer fix point theorems] and to interpret [his] combinatorial lemma [on effectively labelling a restricted simplex] as an example of the non-retraction theorem is incongruous. This is because Scarf, too, like the Brouwer at the time of the original proof of his fix-point theorem, uses the full paraphernalia of non-constructive logical principles to link the Brouwer and non-retraction theorems and his combinatorial lemma[24].

The Kakutani fixed point theorem (Theorem 1 in [30]), and Kakutani's Min-Max Theorem (Theorem 3, ibid). These two theorems, in turn, invoke Theorem 2 and the Corollary (ibid, p.458), which are based on Theorem 1 (ibid, p. 457). This latter theorem is itself based on the validity of the Brouwer fixed point theorem, which is Non-constructifiable (cf.,[11]).

2.3 Scarf's Fixed Point Algorithm is Non-Constructive

The economic foundations of **CGE** models lie in *Uzawa's Equivalence Theorem* ([**?**], [19], p.719, ff); the mathematical foundations are underpinned by *topological fix point theorems* (Brouwer, Kakutani, etc.). The claim that such models are

[23] Just for ease of reading the discussion in this section I state, here, the simplest possible statement of this theorem:

> **Bolzano-Weierstrass Theorem:** Every bounded sequence contains a convergent subsequence

[24] Scarf uses, in addition, proof by contradiction where, implicitly, LEM (*tertium non datur*) is also invoked in the context of an infinitary instance (cf. [57], pp. 1026-7).

computable or *constructive* rests on mathematical foundations of an algorithmic nature: i.e., on recursion theory or some variety of constructive mathematics. It is a widely held belief that CGE models are both *constructive* and *computable.* That the latter property is held to be true of **CGE** models is evident even from the generic name given to this class of models; that the former characterization is a feature of such models is claimed in standard expositions and applications of **CGE** models. For example in the well known, and pedagogically elegant, textbook by two of the more prominent advocates of applied **CGE** modelling in policy contexts, John Shoven and John Whalley ([61]), the following explicit claim is made:

> "The major result of postwar mathematical general equilibrium theory has been to demonstrate the existence of such an equilibrium by showing the applicability of mathematical fixed point theorems to economic models. ... Since applying general equilibrium models to policy issues involves computing equilibria, these fixed point theorems are important: It is essential to know that an equilibrium exists for a given model before attempting to compute that equilibrium. .....
>
> ...
>
> The weakness of such applications is twofold. First, they provide *non-constructive rather than constructive proofs of the existence of equilibrium*; that is, they show that equilibria exist but do not provide techniques by which equilibria can actually be determined. Second, existence per se has no policy significance. .... Thus, fixed point theorems are only relevant in testing the logical consistency of models prior to the models' use in comparative static policy analysis; such theorems do not provide insights as to how economic behavior will actually change when policies change. *They can only be employed in this way if they can be made constructive* (i.e., be used to find actual equilibria). *The extension of the Brouwer and Kakutani fixed point theorems in this direction is what underlies the work of Scarf .... on fixed point algorithms ....*"
>
> ibid, pp12, 20-1; italics added

Quite apart from a direct implication of the results of the previous subsection falsifying the above claims, they are also untenable because the Uzawa Equivalence Theorem is provably undecidable. This is the topic of the next subsection.

2.4 The Uzawa Equivalence Theorem

The Uzawa Equivalence theorem is the fulcrum around which the *theory* of **CGE** modelling revolves. This key theorem[25] provides the theoretical justifica-

[25] To the best of my knowledge, none of the standard advanced textbooks in mathematical economics, microeconomics or general equilibrium theory (Kreps, Varian, etc.), except the

tion for relying on the use of the algorithms that have been devised for determining general economic equilibria as fix points using essentially non-constructive topological arguments. The essential content of the theorem is the mathematical equivalence between a precise statement of *Walras' Existence Theorem* (**WET**) and Brouwer's (or any other relevant) Fix-Point Theorem. To study the algorithmic - i.e., computable and constructive - content of the theorem, it is necessary to analyse the assumptions underpinning **WET**, the nature of the proof of economic equilibrium existence in **WET** and the nature of the proof of equivalence. By the 'nature of the proof' I mean, of course, the constructive content in the logical procedures used in the demonstrations- whether, for example, the law of double negation or the law of the excluded middle (*tertium non datur*) is invoked in non-finitary instances. Therefore, I shall, first, state an elementary version of **WET** (cf., [79], p. 60 or [71], p. 136).

Theorem 16 *Walras" Existence Theorem (**WET**)*

Let the excess demand function, $X(p) = [x_1(p), ..., x_n(p)]$*, be a mapping from the price simplex,* S*, to the* $\mathbb{R}^N$

commodity space; i.e., $X(p) : S \to \mathbb{R}^N$

where:

i). $X(p)$ *is continuous for all prices,* $p \in S$

ii). $X(p)$ *is homogeneous of degree* 0*;*

iii). $p.X(p) = 0, \forall p \in S$ *(Walras' Law holds:* $\sum_{i=1}^{n} p_i x_i(p) = 0,\ \forall p \in S$*)*[26]

Then:

$\exists p^* \in S$*, s.t.,* $X(p^*) \leq 0$*, with* $p_i^* = 0$*,* $\forall i$*, s.t.,* $X_i(p^*) < 0$

The finesse in this half of the equivalence theorem, i.e., that **WET** implies the Brouwer fix point theorem, is to show the feasibility of devising[27] a continuous excess demand function, $X(p)$, satisfying Walras' Law (and homogeneity), from an arbitrary continuous function, say $f(.) : S \to S$, such that the equilibrium price vector implied by $X(p)$ is also the fix point for $f(.)$, from which it is 'constructed'. The key step in proceeding from a given, arbitrary, $f(.) : S \to S$ to an excess demand function $X(p)$ is the definition of an appropriate scalar:

two by Cornwall ([18]) and Starr ([71]), even refer to Uzawa's theorem.

[26] As far as possible I attempt to retain fidelity to Uzawa's original notation and structure, even although more general formulations are possible. .

[27] I have to seek recourse to words such as 'devise' to avoid the illegitimate use of mathematically loaded terms like 'construction', 'choice', 'choose', etc., that the literature on **CGE** modelling is replete with, signifying, illegitimately, possibilities of meaningful – i.e., algorithmic – 'construction', 'choice', etc. For example, Uzawa, at this point, states: "We *construct* an excess demand function.." (op.cit, p.61; italics added; Starr, at a comparable stage of the proof states: "If we have *constructed* [the excess demand function] cleverly enough..." (op.cit., p.137; italics added). Neither of these claims are valid from the point of view of any kind of algorithmic procedure.

$$\mu(p) = \frac{\sum_{i=1}^{n} p_i f_i[\frac{p}{\lambda(p)}]}{\sum_{i=1}^{n} p_i^2} = \frac{p.f(p)}{|p|^2} \tag{2}$$

Where:

$$\lambda(p) = \sum_{i=1}^{n} p_i \tag{3}$$

From (1) and (2), the following excess demand function, $X(p)$, is defined:

$$x_i(p) = f_i(\frac{p}{\lambda(p)}) - p_i \mu(p) \tag{4}$$

i.e.,

$$X(p) = f(p) - \mu(p)p \tag{5}$$

It is simple to show that (3) [or (4)] satisfies (i)-(iii) of Theorem 3 and, hence, $\exists p^*$ s.t., $X(p^*) \leq 0$ (with equality unless $p^* = 0$). Elementary (non-constructive) logic and economics then imply that $f(p^*) = p^*$. I claim that the procedure that leads to the definition of (3) [or, equivalently, (4)] to determine $p^*$is provably *undecidable*. In other words, the crucial scalar in (1) cannot be defined recursion theoretically (and, *a fortiori*, constructively) to effectivize a sequence of projections that would ensure convergence to the equilibrium price vector.

Theorem 17 *$X(p^*)$, as defined in (3) [or (4)] above is undecidable; i.e., cannot be determined algorithmically.*

Proof. *Suppose, contrariwise, there is an algorithm which, given an arbitrary $f(.) : S \to S$, determines $X(p^*)$. This means, therefore, in view of (i)-(iii) of Theorem 1, that the given algorithm determines the equilibrium p^* implied by* ***WET****. In other words, given the arbitrary initial conditions $p \in S$ and $f(.) : S \to S$, the assumption of the existence of an algorithm to determine $X(p^*)$ implies that its* halting *configurations are decidable. But this violates the* undecidability of the Halting Problem for Turing Machines*. Hence, the assumption that there exists an algorithm to determine - i.e., to construct - $X(p^*)$ is untenable.* ■

Remark 18 *The algorithmically important content of the proof is the following. Starting with an arbitrary continuous function mapping the simplex into itself and an arbitrary price vector, the existence of an algorithm to determine $X(p^*)$ entails the feasibility of a procedure to choose price sequences in some determined way to check for p^* and to halt when such a price vector is found. Now, the two scalars, μ and λ are determined once $f(.)$ and p are given. But an arbitrary initial price vector p, except for flukes, will not be the equilibrium price vector p^*. Therefore the existence of an algorithm would imply that there is a systematic procedure to choose price vectors, determine the values of $f(.)$, μ and λ and the* *associated excess demand vector $X(p;\mu,\lambda)$. At each determination of such an excess demand vector, a projection of the given, arbitrary, $f(p)$, on the current $X(p)$, for the current p, will have to be tried. This procedure must continue till the projection for a price vector results in excess demands that vanish for some price. Unless severe recursive constraints are imposed on price sequences - constraints make very little economic sense - such a test is algorithmically infeasible. In other words, given an arbitrary, continuous, $f(.)$, there is no procedure - algorithm (constructive or recursion theoretic) - by which a sequence of price vectors, $p \in S$, can be systematically tested to find p^*.*

Remark 19 *In the previous remark, as in the discussion before stating Theorem 4, I have assumed away the difficulties with uncomputable functions, prices and so on. They simply add to complications without changing the nature of the content of Theorem 4.*

2.5 *Negishi's Method* is Non-Constructive

> "The *method of proof* used in this essay [i.e., in [39]] has been found useful also for such problems as equilibrium in infinite dimensional space and *computation of equilibria*."
>
> [41], p. xiv; italics added.

What exactly was Negishi's *method of proof* and how did it contribute to the *computation of equilibria*?

A pithy characterisation of the difference between the standard approach to *proving the existence* of an Arrow-Debreu *equilibrium*, and its *computatio*n by a tâtonnement procedure – i.e., algorithm – of a mapping from the price simplex to itself, and the alternative *Negishi method* of iterating the weights assigned to individual utility functions that go into the definition of a social welfare function which is maximised to determine – i.e., compute – the equilibrium, captures the key innovative aspect of the latter approach. Essentially, therefore, the difference between the standard approach to the proof of existence of equilibrium Arrow-Debreu prices, and their computation, and the *Negishi approach* boils down to the following:

- The standard approach proves the existence of Arrow-Debreu equilibrium prices by an appeal to a fixed point theorem and computes them – the equilibrium prices – by invoking the *Uzawa equivalence theorem* ([**?**]) and devising an algorithm for the excess demand functions that map a price simplex into itself to determine the fixed point ([56]).

- The Negishi approach *proves*, given initial endowments, *the existence* of individual welfare weights defining a social welfare function, whose *maximization* (subject to the usual constraints) *determines* the identical Arrow-Debreu *equilibrium*. The standard mapping of excess demand functions, mapping a price simplex into itself to determine a fixed point, is replaced by a mapping from the space of utility weights into itself, appealing to the same kind of fixed point theorem (in this case, the Kakutani fixed point theorem) to prove the existence of equilibrium prices.

- In other words, the method of proof of existence of equilibrium prices in the one approach is replaced by the *proof of existence* of 'equilibrium utility weights', both appealing to traditional *fixed point theorems* ([**?**], [**?**], and [30][28]).

- In both cases, the computation of equilibrium prices on the one hand and, on the other, the computation of equilibrium weights, algorithms are devised that are claimed to determine (even if only approximately) the same fixed points.

Before proceeding any further, I should add that I am in the happy position of being able to refer the interested reader to a scholarly survey of Negishi's work. Takashi Negishi's outstanding 'contributions to economic analysis' are brilliantly and comprehensively surveyed by Warren Young in his recent paper ([95]).

However, no one – to the best of my knowledge – has studied Negishi's *method of proof* from the point of view of *constructivity* and *computability*. Young's perceptive - and, in my opinion, entirely correct - identification of the crucial role played by Negishi (1960) in 'both "theoretical" and "applied" research program in general equilibrium analysis' is, in fact, about *methods of **existence** proofs* and *computable general equilibrium* (**CGE**), and its offshoots, in the form of *applied computable general equilibrium analysis* **ACGE**) – even leading up to current frontiers in computational issues in *DSGE* models (cf., [29], pp. 52-57, for example).Now, it is generally agreed that the *Negishi method of existence proof* is an applications of fixed point theorems on the *utility simplex*, in contrast to the 'standard' way of applying such theorems to the *price simplex* (cf., [13], p. 138, and above).

There are two theorems in [39]. I shall concentrate on *Theorem 2* (ibid, p.5), which (I think) is the more important one and the one that came to play the important role justly attributed to it via the *Negishi Research Program* outlined by Young (op.cit)[29].

Proposition 20 *The Proof of the Existence of Maximising Welfare Weights in the Negishi Theorem is Nonconstructive*

Remark 21 *Negishi's proof relies on satisfying the Slater (Complementary)*

[28] There is a curious – albeit inessential – 'typo' in Negishi's reference to Kakutani's classic as having been published in 1948. The 'typo' is not 'corrected' even in the reprinted version of [39] in [41].

[29] To demonstrate the *nonconstructive* elements of Theorem 1 (*ibid*, p.5), I would need to include almost a tutorial on constructive mathematics to make clear the notion of *compactness* that is *legitimate in constructive analysis*. For reasons of 'readability' and 'deeper' reasons of aesthetics and mathematical philosophy, I shall refer to my two main results as 'Propositions' and their plausible validity as 'Remarks', and not as 'Theorems' and 'proofs', respectively.

Slackness Conditions ([69][30]). Slater's proof[31] of these conditions invoke the Kakutani fixed point theorem (Theorem 1 in [30]), and Kakutani's Min-Max Theorem (Theorem 3, ibid). These two theorems, in turn, invoke Theorem 2 and the Corollary (ibid, p.458), which are based on Theorem 1 (ibid, p. 457). This latter theorem is itself based on the validity of the Brouwer fixed point theorem, which is Non-constructifiable (cf., [11]).

Proposition 22 *The vector of maximising welfare weights, derived in the Negishi Theorem, is uncomputable*

Remark 23 *A straightforward implication of Claim 1*

Discovering the exact nature and source of appeals to nonconstructive modes of reasoning, appeals to undecidable disjunctions and reliance on nonconstructive mathematical entities in the formulation of a theorem is a tortuous exercise. The nature of the pervasive presence of these three elements – i.e., nonconstructive modes of reasoning, primarily the reliance on *tertium non datur*, undecidable disjunctions and nonconstructive mathematical entities – in any standard theorem and its proof, and the difficulties of discovering them, is elegantly outlined by Fred Richman ([52], p. 125; italics added):

> "Even those who like algorithms have remarkably little appreciation of the thoroughgoing algorithmic thinking that is required for a constructive proof. This is illustrated by the nonconstructive nature of many proofs in books on numerical analysis, the theoretical study of practical numerical algorithms. I would guess that most realist mathematicians are unable even to recognize when a proof is constructive in the intuitionist's sense.
>
> *It is a lot harder than one might think to recognize when a theorem depends on a nonconstructive argument. One reason is that proofs are rarely self-contained, but depend on other theorems whose proofs depend on still other theorems. These other theorems have often been internalized to such an extent that we are not aware whether or not nonconstructive arguments have been used, or must be used, in their proofs.* Another reason is that the law of excluded middle [LEM] is so ingrained in our thinking that we do not distinguish between different formulations of a theorem that are trivially equivalent given LEM, although one formulation may have a constructive proof and the other not."

[30] This classic by Slater must easily qualify for inclusion in the class of pioneering articles that remained forever in the '*samizdat*' status of a *Discussion Paper*!

[31] I should add that the applied general equilibrium theorists who use Negishi's method to 'compute' (uncomputable) equilibria do not seem to be fully aware of the implications of some of the key assumptions in Slater's complementary slackness conditions. That Negishi ([39]) is aware of them is clear from his *Assumption 2* and *Lemma 1*.

2.6 *Classical* Computable Economics

> "The method of 'postulating' what we want has many advantages; they are the same as the advantages of theft over honest toil. Let us leave them to others and proceed with our honest toil."
>
> Bertrand Russell ([54], p. 71)

In computable economics, as in any computation with analogue computing machines or in classical behavioural economics, all solutions are based on *effectively computable* methods. Thus computation is intrinsic to the subject and all formally defined entities in computable economics – as in classical behavioural economics – are, therefore, algorithmically grounded.

Given the algorithmic foundations of computability theory and the intrinsic dynamic form and content of algorithms, it is clear that this will be a '*mathematics with dynamic and algorithmic overtones*'[32]. This means, thus, that computable economics is a case of a new kind of mathematics in old economic bottles. The 'new kind of mathematics' implies new questions, new frameworks, new proof techniques - all of them with algorithmic and dynamic content for digital domains and ranges.

Some of the key formal concepts of computable economics are, therefore: *solvability & Diophantine decision problems*, *decidability* & *undecidability*, *computability & uncomputability*, *satisfiability*, *completeness* & *incompleteness*, *recursivity* and *recursive enumerability*, *degrees of solvability* (*Turing degrees*), *universality* & the *Universal Turing Machine* and *Computational*, *algorithmic* and *stochastic complexity*. The proof techniques of computable economics, as a result of the new formalisms, will be, typically, invoking methods of: *Diagonalization*, *The Halting Problem for Turing Machines*, *Rice's Theorem*, *Incompressibility theorems*, *Specker's Theorem*, *Recursion Theorems.* For example, the *recursion theorems* will replace the use of traditional, non-constructive and uncomputable, topological fix point theorems, routinely used in orthodox mathematical analysis. The other theorems have no counterpart in non-algorithmic mathematics.

In the spirit of pouring new mathematical wines into old economic bottles, the kind of economic problems of a digital economy that computable economics is immediately able to grant a new lease of life are the classic ones of: computable and constructive existence and learning of rational expectations equilibria, computable learning and complexity of learning, computable and bounded rationality, computability, constructivity and complexity of general equilibrium models, undecidability, self-reproduction and self-reconstruction of models of economic dynamics (growth & cycles), uncomputability and incompleteness in (finite and

[32]

> "I think it is fair to say that for the main existence problems in the theory of economic equilibrium, one can now bypass the fixed point approach and attack the equations directly to give existence of solutions, with a simpler kind of mathematics and *even mathematics with dynamic and algorithmic overtones*."
>
> [70], p.290; italics added.

infinite) game theory and of Nash Equilibria,decidability (playability) of arithmetical games, the intractability (computational complexity) of optimization operators; etc.

Suppose the starting point of the computable economist whose visions of actual economic data, and its generation, are the following:

Conjecture 24 *Observable variables are sequences that are generated from recursively enumerable but not recursive sets, if rational agents underpin their generation.*

An aside: In 1974 Georg Kreisel posed the following problem:

> "We consider theories, ... and ask if every sequence of natural numbers or every real number which is well defined (*observable*) *according to the theory* must be recursive or, more generally, *recursive in the data*. .... Equivalently, we may ask whether any such sequence of numbers, etc., can also be generated by an ideal computing or Turing Machine if the data are used as input. The question is certainly not empty because most objects considered in a ... theory are not computers in the sense defined by Turing. ...."
>
> [34], p.11

The above conjecture has been formulated after years of pondering on Kreisel's typically thought-provoking question. More recently, a reading of Osborne's stimulating book ([46]), was also a source of inspiration in the formulation of the conjecture as an empirical disciplining criterion for computable economics.

The conjecture is also is akin to the orthodox economic theorist and her handmaiden, the econometrician, assuming that all observable data emanate from a structured probability space and the problem of inference is simply to determine, by statistical or other means the parameters that characterise their probability distributions.

All the way from microeconomic supply and demand functions to monetary macroeconomic variables, parameters and functions, *Diophantine relations, equations and functions predominate in computable economics.* This is because the natural data types in economics are, at best, rational numbers. Hence, the following famous theorem is used extensively.

Theorem 25 ***Undecidability of Hilbert's tenth problem***

In computable economics the path towards a pedagogical presentation of this classic result if via an exposition of Rabin's Theorem.

The following four theorems are used to prove the uncomputability of rational expectations equilibria in orthodox frameworks and to construct computable rational expectations equilibria in computable macroeconomics, respectively.

Theorem 26 ***Rice's Theorem****: Let C be a class of partial recursive functions. Then C is not recursive unless it is the empty set, or the set of all partial recursive functions.*

Theorem 27 ***Fix Point Theorem***

Suppose that $\Phi : \mathcal{F}_m \longrightarrow \mathcal{F}_n$ *is a recursive operator (or a recursive program Þ). Then there is a partial function* f_ϕ *that is the least fixed point of* $\Phi :$

Theorem 28 $\Phi\left(f_\phi\right) = f_\phi$;
If $\Phi\left(g\right) = g$, *then* $f_\phi \sqsubseteq g$.

Remark 29 *If, in addition to being partial,* f_ϕ *is also total, then it is the* ***unique least fixed point.***

Finally, related to invariance theorems in the domain of algorithmic complexity theory and the fix point theorem of classical recursion theory, we have the *recursion theorem*, essential for understanding self-reproduction and self-reconstruction (for computable growth theory):

Theorem 30 ***Recursion Theorem*** *Let* T *be a Turing Machine that computes a function:*

$$t : \Sigma^* \times \Sigma^* \longrightarrow \Sigma^* \tag{6}$$

Then, there is a Turing Machine R *that computes a function*:

$$r : \Sigma^* \longrightarrow \Sigma^* \tag{7}$$

such that, $\forall \omega$:

$$r(\omega) = t\left(\langle R\rangle, \omega\right) \tag{8}$$

where,
$\langle R\rangle$: denotes the encoding of the Turing Machine into its standard representation as a bit string;
and the $*$(*star*) operator denotes its standard role as a *unary operator* defined as: $A^* = \{x_1, x_2, ..., x_k \mid k \geq 0, \forall x_i \in A\}$

The idea behind the *recursion theorem* is to formalize the activity of a Turing Machine that can obtain its own description and, then, compute with it. This theorem is essential, too, for formalizing, recursion theoretically, a model of *growth* in a digital economy and to determine and learn, computably and constructively, *rational expectations equilibria.* The *fix point theorem* and the *recursion theorem* are also indispensable in the computable formalization of *policy ineffectiveness* postulates, *time inconsistency* and *credibility* in the theory of macroeconomic policy. Even more than in microeconomics, where topological fix point theorems have been indispensable in the formalizations underpinning existence proofs, the role of the above *fix point theorem* and the related *recursion theorem* are absolutely fundamental in what I come to call *Computable Macroeconomics.*

Anyone who is able to formalize these theorems, corollaries and conjectures and work with them, would have mastered some of the key elements that form the core of the necessary mathematics of computable economics. Unlike so-called computable general equilibrium theory and its offshoots, computable economics – and *its* offshoots – are intrinsically computational and numerical.

3 Computable Economics: Towards the Frontiers

"These proofs necessarily involve the use of *number theory – a branch of mathematics unfamiliar to most economists.*"

Clower & Howitt, [15], footnote 3, p. 452; italics added.

Rózsa Péter's opening lines in her classic text on **Recursive Functions** ([50], p. 7; italics added) must only reinforce this hopelessness of the Computable Economist:

> "*The theory of recursive functions properly belongs to number theory*; indeed, the theory of recursive functions is, so to speak, the function theory of number theory. ... The notion of recursive function marks off those functions whose values can be effectively calculated at every particular point; and *just those functions are useful in the natural sciences.* Though the variables of recursive functions do not run through all real numbers but only the natural numbers, probability theory as well as quantum theory operates with functions of this latter kind; and recently recursive functions have begun to be applied in analysis too."

If 'the theory of recursive functions properly belongs to number theory' and if number theory is 'a branch of mathematics *unfamiliar* to most economists', then what hope is their that economists would be familiar with recursive function theory?

At least since Walras devised the *tâtonnement* process and Pareto's appeal to the market as a computing device, there have been sporadic attempts to find mechanisms to solve a system of supply-demand equilibrium equations, going beyond the simple counting of equations and variables. But none of these attempts to devise mechanisms to solve a system of equations were predicated upon the elementary fact that the data types – the actual numbers – realised in, and used by, economic processes were, at best, rational numbers (see the above observation on 'natural sciences' in the quote from Rózsa Péter's book on **Recursive Functions,** and also [46]). The natural equilibrium relation between supply and demand, respecting the elementary constraints of the equally natural data types of market – or any other kind of economy – should be framed as a Diophantine decision problems, and the way arithmetic games are formalised and shown to be effectively unsolvable in analogy with the *Unsolvability* of *Hilbert's Tenth Problem* (cf. [37]).

The Diophantine decision theoretic formalization is, thus, common to at least three kinds of computable economics: classical behavioural economics, algorithmic game theory in its incarnation as arithmetic game theory and elementary equilibrium economics. Even those, like Smale ([70]), who have perceptively discerned the way the problem of finding mechanisms to solve equations was subverted into formalizations of inequality relations which are then solved by appeal to (unnatural) non-constructive, uncomputable, fixed point theorems did not go far enough to realise that the data types of the variables and parameters entering the equations needed not only to be constrained to be non-negative, but also to be rational (or integer valued). Under these latter constraints, economics in its behavioural, game theoretic and microeconomic modes must come to terms with *absolutely (algorithmically) undecidable problems*. This is the cardinal message of the path towards computable economics.

Therefore, if orthodox algorithmic game theory, orthodox mechanism theory and computable general equilibrium theory have succeeded in computing their respective equilibria, then they would have to have done it with algorithms that are not subject to the strictures of the *Church-Turing Thesis* or do not work within the (constructive) proof-as-algorithm paradigm. This raises the mathematical meaning of the notion of algorithm in algorithmic game theory, orthodox mechanism theory and computable general equilibrium theory (and varieties of so-called computational economics). Either they are of the kind used in numerical analysis and so-called 'scientific computing' (as if computing in the recursion and constructive theoretic traditions are not 'scientific'; see [10] for a lucid definition and discussion of this seemingly innocuous concept) and, if so, their algorithmic foundations are, in turn, constrained by either the *Church-Turing Thesis* (as in [8]) or the (constructive) proof-as-algorithm paradigm; or, the economic system and its agents and institutions are computing the formally uncomputable and deciding the algorithmically undecidable (or are formal systems that are inconsistent or incomplete).

I believe Goodstein's algorithm, [25] could be the paradigmatic example for modelling rational - or integer - valued algorithmic (nonlinear) economic dynamics (see, for example, [49]). Every sense in which the notion of algorithm has been discussed above, for the path towards computable economics, is most elegantly satisfied by this line of research, a line that has by-passed the mathematical economics and nonlinear macrodynamics community. This is the only way I know to be able to introduce the algorithmic construction of an integer-valued dynamical system possessing a very simple global attractor, and with immensely long, effectively calculable, transients, whose existence is unprovable in *Peano Arithmetic*. Moreover, this kind of nonlinear dynamics, subject to *SSID*, ultra-long transients and possessing simple global attractors whose existence can be encapsulated within a classic *Gödelian, Diophantine*, decision theoretic framework, makes it also possible to discuss effective policy mechanisms (cf. [32]).

Kreisel's characteristically perceptive observation (see quote above, in the previous section), a plea for understanding the way to use the 'Goodstein algorithm' in economic dynamics and the economist's penchant for drawing curves and for working with numbers defined over the real numbers, convinces me that the most important frontier for computable economics is *computable analysis,* ([92]; coming down the [6] tradition) or *computable calculus* ([1], where a judicious combination of constructive logic and recursion theory is used). I have come to believe that every mathematically minded economist should be familiar with the *graph theorem of classical recursion theory* ([45], p. 135-6), and not simply be bamboozled by the Dirichlet-Kuratowski graph concept. The interaction between *recursive* and *recursively enumerable* sets, computable functions and *functions 'plottable' on a digital computer's screen* should be made clear to all students of economics, almost more importantly than teaching them probability theory, statistics and the like. This is implicit in some of the claims about the notion and definition of computation universality I have routinely been using in classical computable economics.

With an integration of *classical recursion theory* (using, say, [17]), *computable analysis* and a familiarity with the framework of *Diophantine Decision Problems*, *classical* computable economics will be ready to embark on the path towards *modern* computable economics, where not only the *theory* of the computer will be an underpinning of economic theory; but also the empirical use of the hardware, the pixels and the resolution that make the screen as much a part of the computable economist's 'box of tools' as its theory, will enrich the experiences of being educated to be a computable economist.

A decade ago, after reading my first book on *Computable Economics* ([82]), Herbert Simon wrote, on 25 May, 2000, to one of my former colleagues as follows:

> "I think the battle has been won, at least the first part, although it will take a couple of academic generations to clear the field and get some sensible textbooks written and the next generation trained."

The 'battle' that 'had been won' against orthodox , non-algorithmic, economic theory had taken Simon almost half a century of sustained effort in making Classical Behavioural Economics and its algorithmic foundations the centrepiece of his research at the theoretical frontiers of computational cognitive science, behavioural economics, evolutionary theory and the theory of problem solving. Yet, hel felt more time was needed. For the full impact of a computable approach to economics, I am not sure orthodoxy will permit 'the clearing of the field', even if 'sensible textbooks' are written to get the 'next generation trained'. All the same, it is incumbent upon us to make the attempt to prepare for a 'computable and constructive' future, by writing the 'sensible textbooks' for the next – or future – generations of students, who will be the harbingers of the computable approach to economics.

Samuel Beckett, in the opening sentence of his masterly essay on *Proust* ([7]), summarised, with characteristically subtle depth and brevity, the whole philosophy of a computable economics that aims to be underpinned by Husserl's phenomenology (italics added):

> "The Proustian equation is never simple. The unknown, choosing its weapons from a hoard of values, is also the *unknowable.*"

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