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PRODUCTION FUNCTIONS BEHAVING BADLY RECONSIDERING FISHER AND SHAIKH

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Abstract

We reconsider Anwar Shaikh's critique of the neoclassical theory of growth and distribution based on its use of aggregate production functions. This is done by reconstructing and extending Franklin M. Fisher's 1971 computer simulations, which Shaikh used to support his critique. Together with other recent extensions to Shaikh's seminal work, the results support and strengthen the evidence against the use of *neoclassical* aggregate production functions.

Keywords: HUMBUG Production Function, Cobb-Douglas Production Function, Aggregation, Computational Techniques.

JEL Classifications: C61, C63, C 67, O 47.

1 Introduction

The notion of aggregate production functions has long been a widely used theoretical concept in economics and remains among the fundamental concepts presented in almost every course in micro and macroeconomics. Furthermore, aggregate production functions constitute the core of the supply side in most modern econometric as well as theoretical models, e.g., *CGE* models.

In 1974, Anwar Shaikh proposed a serious critique of the neoclassical theory of growth and distribution based on its use of aggregate production functions. Empirical studies had hitherto shown that aggregate production functions of the Cobb-Douglas type usually fit the data well, and that the estimated coefficients typically coincide with observed wage and profit shares of income. These empirical findings were used not only to support the neoclassical theory of growth and distribution, but also to contest non-micro founded theory, because of its lack of this kind of "indisputable" support.

However, as Shaikh (1986, p. 191) claims the 'apparent empirical strength of aggregate production functions is often interpreted as support for neoclassical theory. But there is neither theoretical nor empirical basis for this conclusion.'

The purpose of this chapter is to reconsider Shaikh's critique (Shaikh, 1974; see also Shaikh, 1980, 1986, 2005) and parts of the subsequent work on the subject. This is done by reconstructing and extending the original computer simulations by Fisher (1971), which Shaikh used to support his thesis. We extend Fisher's simulations by introducing CES production functions at the industry level, but continue to estimate a simple Cobb-Douglas production function from the aggregated data. This, we claim, provides further insight into the extent and implications of Shaikh's critique.

We will show that Fisher's simulation experiment can be reconstructed and by doing this we can also confirm Fisher's 1971 findings, which in itself is of interest because these results have been widely used, but to the best of our knowledge never verified. Furthermore, by inspecting the goodness of fit in Fisher's almost 1000 experiments, we will show that Shaikh's interpretation of Fisher's work is correct. Finally, we compare our results with those obtained by McCombie and Dixon (1991), Felipe and Holz (2001), and Shaikh (2005). In line with these researchers, we find evidence to support a more general version of Shaikh's original critique.

Section 2 and 3 present Shaikh's original critique and subsequent extensions, Section 4 and 5 deals with Fisher's original model and the reconstruction, and Section 6 presents the extension of Fisher's model. Section 7 concludes the paper with a discussion on the consequences of this critique for the neoclassical theory of growth and distribution.

2 Laws of Algebra

Shaikh claims and proves that whenever input–output data exhibit constant income shares, there is a very good chance that regardless of the true nature of the data, an aggregate production function of the Cobb-Douglas type will fit the data very well. Therefore, Shaikh concludes that when one estimates a Cobb-Douglas production function on input–output data, there is a good chance that one only observes *laws of algebra* and not *laws of production*.

The following is a concise version of Shaikh's proof. It starts with the universal income accounting identity, *viz.*

$$Y = wL + rK \tag{2.1}$$

Let $y = Y/L,\ k = K/L,\ \alpha = rK/Y,\ 1 - \alpha = wL/Y$, and assumes that labour's share of income is constant over time. Now (2.1) can be written as $y = w + rk$.

$$\begin{aligned} y = w + rk \Rightarrow \dot{y} = \dot{w} + \dot{r}k + r\dot{k} \Leftrightarrow \frac{\dot{y}}{y} &= \frac{w}{y}\frac{\dot{w}}{w} + \frac{rk}{y}\frac{\dot{r}}{r} + \frac{rk}{y}\frac{\dot{k}}{k} \\ \Rightarrow \frac{\dot{y}}{y} &= (1-\alpha)\frac{\dot{w}}{w} + \alpha\frac{\dot{r}}{r} + \alpha\frac{\dot{k}}{k} \\ \Rightarrow \ln y &= (1-\alpha)\ln w + \alpha \ln r + \alpha \ln k + \ln c_0 \\ \Rightarrow y &= C_1 k^{\alpha} \Leftrightarrow Y = C_1 K^{\alpha} L^{1-\alpha} \end{aligned} \tag{2.2}$$

Where the shift term C_1 is given by:

$$C_1 = c_0 \cdot r^{\alpha} w^{1-\alpha} \tag{2.3}$$

To sum up, from a tautology of input–output data and an assumption of constant input shares (plus an implicit assumption of differentiable functions), a function of the Cobb-Douglas type follows directly through basic applications of the laws of algebra! This is an important result, since it implies that regressions of a Cobb-Douglas production function, given that the data exhibit constant input shares, are predetermined to give high correlation coefficients, and are thereby meaningless.

Because of this Shaikh named the Cobb-Douglas production function the "HUMBUG" production function, and emphasized the message by showing that the coordinates in the Cartesian plane spelling the word "HUMBUG" together with profit shares from the US (Solow's 1957 data) could be fitted almost perfectly by a Cobb-Douglas production function (Shaikh, 1974).[1]

[1] Shaikh's results have been challenged by Solow (1974), but subsequently defended by Shaikh (1980), after which the discussion, to the best of our knowledge, seems to have gone quiet.

3 Related Work

The use of aggregate production functions has long been a subject of serious discussion, and no consensus has yet been reached. The debate can be divided into two major parts: the so-called *index number problem* and *value problem*, which respectively refer to the problems of aggregation and the logical problem in determining the value of capital independently of the profit rate. Here we deal only with the index number problem, or to be more specific, the issues of interpreting aggregated empirical results from technologically diverse economies.[2]

Following the first paper by Shaikh on the HUMBUG production function, a number of theoretical and empirical studies have been published on the subject with J. Felipe and J.S.L. McCombie as the main contributors (see Felipe and Fisher, 2003).

However, the papers by McCombie and Dixon (1991), Felipe and Holz (2001), and Shaikh (2005) are of special interest for this article. McCombie and Dixon (1991) prove that Shaikh's critique also stands when factor shares are not constant as long as the shift term grows with a constant rate. Furthermore, they show that even if the shift term does not grow with a constant rate, it is possible 'with sufficient ingenuity, to find a functional form which will produce a very good fit to the underlying identity' McCombie and Dixon (1991, p. 40), and they refer to the CES and the translog production function as potential candidates.[3]

The paper by Felipe and Holz (2001, p. 281) presents an interesting Monte Carlo simulation that shows

> that the Cobb-Douglas form is robust to relatively large variations in the factor shares. However, what makes this form quite often fail are the variations in the growth rates of the wage and profit rates. The weighted average of these two growth rates has been shown to be the coefficient of the time trend. This implies that, in most applied work, a Cobb-Douglas form (i.e. approximation to the income accounting identity) should work. We just have to find *which* Cobb-Douglas form with a dose of patience in front of the computer.

Moreover, they show that spurious regression cannot explain the systematic (near) perfect fit of the Cobb-Douglas function.

In a recent paper, Shaikh (2005) presents a more general version of his original results; the so-called *Perfect Fit Theorem*. This theorem states that, given a stable labour share, it is always possible to construct a time function $F(t)$, 'that will always make fitted production functions work "perfectly" in the sense of Solow: that is, make them yield perfect econometric fits with partial derivatives that closely

[2]See Cohen and Harcourt (2003) for a extensive survey and Zambelli (2004) for a more concise survey on the value problem.

[3]See also Felipe and McCombie (2001) for a very interesting study of the CES production function's ability to fit input–output data, where they reconsider Arrow *et. al.* (1961) seminal work on the CES function.

approximate observed factor prices' Shaikh (2005, p. 457). The time function must merely be constructed in the following way:

$$SR_t = \alpha_{t-1}\Delta \log r + (1 - \alpha_{t-1})\Delta \log w \tag{3.1}$$

$$F_t = \beta + h\left(SR_t - \tfrac{1}{t}\sum SR_t\right) \tag{3.2}$$

Note that (3.1) resembles the shift term (2.3) and that an affine function of the Solow Residual SR_t yields an affine time function F_t.

Coherently with McCombie and Dixon (1991) and Felipe and Holz (2001), we agree that the assumption of constant input shares is not needed for the main results to hold. However, we think that the Cobb-Douglas production function fits the data well even when shares are not stable **and** the shift term (2.3) fails in a test of trend stationarity.

As for Shaikh's Perfect Fit Theorem, we acknowledge the power of the theorem in its ability to ensure a perfect fit, but we also underline that constant input–shares are still a required assumption.

It is important to note that our results are not conditioned on *a perfect fit*, but on *a good fit*, by which we mean a fit that would make most econometricians, given the usual reservations, accept the model as a good description of the data. In other words, we are not *per se* interested in the theoretical — but very possible — possibility of making a neoclassical production function fit the data perfectly, with the help from cleverly constructed trend terms or more flexible functional forms such as the CES or translog. We are merely interested in the basic method of regressing a log-linearized Cobb-Douglas function with a simple (affine) trend term on input–output data, and will show that this method often is sufficient to ensure a good fit, even when the underlying data should not be explainable with such a simple model. We believe this is an interesting approach, because this method is extensively used by not only students of economics, but also established researchers. Showing that these claims hold will be the main quest in the following.

4 Fisher's Model

The purpose of Fisher's 1971 paper was to study the conditions under which the production possibilities of a technologically diverse economy can be represented by an aggregate production function.

The work consists of a huge simulation experiment, where production is simulated at the micro level in a neoclassical model with n heterogeneous firms — all possessing Cobb-Douglas technology. Labour is assumed to be perfectly mobile, but capital and technology are bound to the respective firms. Wage and profit are as usual given by the marginal productivity of labour and capital, respectively. Furthermore, it is assumed that through perfect competition the labour inputs in each period are distributed such that wages would be uniform.

The experiments are divided into two major groups: the so-called *Capital experiments* in which economic development is based on the evolution in the stock of capital, and the *Hicks experiments* in which development is based on changes in a Hicks neutral technology. The experiments were divided into a total of five subgroups depending on the underlying pattern of technological progress. The experiments ran over 20 periods with two, four, or eight firms, and for each experiment three different initial capital or technology endowments, two choices of weights in the production function, and eleven different growth rates in capital or technology were chosen. This gives a total of 990 ($5 \times 3 \times 3 \times 2 \times 11$) unique experiments. See Appendix A for further details.

The experiments were constructed in order to systematically violate the conditions for a theoretically consistent aggregation; see Fisher (1969) for a discussion of these conditions. Capital is aggregated using the profit rates, *viz.*

$$J_t = \sum_{i=1}^{n} \left(\frac{\sum_{t=1}^{20} r_{i,t} K_{i,t}}{\sum_{t=1}^{20} K_{i,t}} \right) K_{i,t} \qquad i = 1, 2, ..., n \quad t = 1, 2, ..., 20 \tag{4.1}$$

The aggregate Cobb-Douglas production function is given by

$$Y_t = A_t J_t^{\alpha} L_t^{1-\alpha} \tag{4.2}$$

4.1 Evaluation of the Model

The primary measurement of performance is the relative root-mean-square error together with the standard deviation of labour's share, *viz.*

$$S = \frac{\sqrt{\frac{1}{20} \sum_{t=1}^{20} (w_t - \hat{w}_t)^2}}{\frac{1}{20} \sum_{t=1}^{20} w_t} \tag{4.3}$$

$$\sigma_\alpha = \sqrt{\frac{1}{20-1} \sum_{t=1}^{20} \left(\hat{\alpha}_t - \frac{1}{20} \sum_{t=1}^{20} \hat{\alpha}_t \right)^2} \tag{4.4}$$

Where $\hat{\alpha}$ and $\hat{w}$ denotes estimated values.

In relation to the analysis of Shaikh's thesis, the standard deviation of labour's share σ_α is important, because of the assumption of a constant labour share.

The parameter α in the aggregate production function is estimated from the following simple log-linearized model:

$$\ln \frac{Y_t}{L_t} = \beta_1 + \beta_2 t + \alpha \ln \frac{J_t}{L_t} + \epsilon_t \tag{4.5}$$

The work presented in Section 3 would predict that the correlation coefficients from the above regression will be equal to or very close to one, whenever the input–output data exhibits either (A) constant factor shares or (B) factor shares that change so that the shift term, see equation (2.3), grows at a constant rate. It is these conditions, we attempt to investigate below.

The trend term $\beta_2 t$ is included to capture what can be characterised as a constant growth in the (aggregated) Hicks neutral technology. Following Fisher (1971, p. 313) this trend term is only included in the Hicks experiments.

To check whether or not assumption A and/or B are satisfied in the experiments, the following methods are used. Constant factor shares are checked by the standard deviation of labour's share σ_α; if this is sufficiently small, it would seem reasonable to accept assumption A. As for assumption B, equation (2.3) states that the shift term is given by a weighted average of the wage and the profit rate; i.e., testing assumption B is equivalent to testing whether or not the following variable is a trend stationary time series.

$$C_t = r_t^{\alpha_t} w_t^{1-\alpha_t} \tag{4.6}$$

Where $1-\alpha_t = \frac{L_t w_t}{Y_t}$ and $r_t = \frac{\sum r_{i,t}}{J_t}$. As usual this is done by including a trend term in the ADF test. Details will be given in the following sections.

5 The Reconstruction

It cannot be expected that the reconstruction yields a perfect replication of Fisher's work, because we do not have information on the pseudo-random number generating algorithm[4].

For reasons of comparability, the original and the reconstructed data are presented in the same type of matrices as Fisher used. These matrices sum up the frequency of observations with a given combination of σ_α and S.

There are some deviations, but these deviations can be justified by the stochastic elements in the model. In any case, Fisher's basic observation is confirmed, i.e., an aggregate production function often provides a good explanation of wages, provided

[4]It is fairly easy to describe the simulations, because The simulation are based on the thorough documentation present in Fisher (1971). We have used MATLAB to write three small programs, which are available upon request.

These programs consist of a master m-file, which basically is Fisher's model as described in his paper plus the extensions. These programs also contain algorithms performing different methods for evaluation, e.g., a set of loops that automatically perform standard ADF tests for stationarity by calculating test statistics and comparing these with the appropriate table values. The significance level for all tests is 5 percent.

The set-up of the experiments is programmed in another m-file, e.g., the different combinations of exogenously given parameter values. Furthermore, this program collects and organises the output.

The last m-file is a wage-equilibrating-algorithm, which is used because in every period in every experiment the wage rates must be uniform among the n firms; see Fisher (1971, p. 308) or Appendix A for further details. The wage-equilibrating-algorithm is extremely time-consuming due to inefficient programming and computational complexity.

S/σ_α	0.0-0.5	0.5-1.0	1.0-1.5	1.5-2.0	2.0-2.5	2.5-3.0	> 3.0
0.0-0.5	296	1	1	0	0	0	0
0.5-1.0	104	91	2	0	0	0	0
1.0-1.5	26	31	41	3	0	0	0
1.5-2.0	13	7	26	20	5	0	0
2.0-3.0	13	11	12	18	19	14	5
3.0-4.0	6	11	4	6	5	2	16
4.0-5.0	3	2	6	3	2	0	13
5.0-10.0	5	14	6	9	7	8	23
10.0-20.0	0	5	6	5	0	4	13
>20.0	1	1	8	5	2	3	26

Table 5.1: *Summary of the Capital and Hicks experiments (original data)*

S/σ_α	0.0–0.5	0.5–1.0	1.0–1.5	1.5–2.0	2.0–2.5	2.5–3.0	> 3.0
0.0–0.5	290	0	0	0	0	0	0
0.5–1.0	122	40	0	0	0	0	0
1.0–1.5	32	50	14	0	0	0	0
1.5–2.0	7	21	37	4	0	0	0
2.0–3.0	24	16	10	17	14	0	0
3.0–4.0	5	10	7	3	6	11	1
4.0–5.0	5	6	12	2	1	2	9
5.0–10.0	14	22	12	9	7	5	15
10.0–20.0	2	11	7	6	8	8	8
>20.0	1	4	10	6	5	7	46

Table 5.2: *Summary of the Capital and Hicks experiments (reconstructed data)*

that the input weights are relatively stable over time. Given Fisher's earlier work on the subject (Fisher, 1969), these results must have been surprising, as the following quote also suggests:

> The point of our results, however, is not that an aggregate Cobb-Douglas fails to work well when labor's share ceases to be roughly constant, it is that an aggregate Cobb-Douglas will continue to work well so long as labor's share continues to be roughly constant ... (Fisher 1971, p. 307)

This reconstruction of Fisher's work allows us to examine the goodness of fit of the underlying regressions. Note that confirming Fisher's original results is *per se* useful, since several authors over the years have referred to these results.

Inspecting the correlation coefficients from Equation 4.5 and standard deviations of labour's share from the 990 unique experiments show that almost all correlation coefficients are very close to 1; 98 percent are greater than 0.90 and 85 percent are greater than 0.99. Moreover, the correlation does not seem to decrease as σ_α increases. Even more interesting, the 96 series with non-trend stationary shift terms continue to give high correlation coefficients: 95 percent of all correlation coefficients are greater than 0.90 and 78 percent are greater than 0.99.

These observations imply that Shaikh's *law of algebra* may very well be more general than formally constrained by assumption A, constant labour shares, and assumption B, a constant growth rate in the shift term.

Note however, that (near) perfect correlation is not always observed, but $R^2 > 0.90$ would lead most researchers, given the usual reservations, to (in this case wrongly) conclude that the estimated model is a good explanation of the underlying system.

To avoid any misconceptions, these results do not contradict those of Shaikh or the subsequent work presented in Section 3, they show that in applied work the risks of making wrong conclusions are not restricted to the cases where assumptions A and B are satisfied.

To ensure that these high correlations are not observations of spurious regressions, the explanatory and the dependent variables in equation (4.5) are checked for possible unit roots by a simple ADF tests. From this it is inferred whether or not there is a potential risk for spurious regression, i.e., if both the dependent and explanatory variables have a unit root. These tests shows that there is only a potential risk for spurious regression in 5.2 percent of the 990 regressions, i.e., the high correlation coefficients cannot be explained by spurious regression. This result is consistent with Felipe and Holz (2001), who also conclude that spurious regression cannot explain the uniformly high fit.

6 The Extended Model

In the following, an extension of Fisher's model is employed to further investigate the generality of Shaikh's critique. The model is changed by replacing the micro Cobb-Douglas production functions with CES production functions, but still estimating an aggregate Cobb-Douglas production function after the conditions for a theoretically consistent aggregation are violated as in the original model.[5] The CES production function is of the following form, where ν is the reciprocal of the elasticity of substitution between capital and labour σ_{KL}, *viz.*

$$y_{i,t} = A_{i,t} \left(\alpha_i K_{i,t}^{1-\nu} + (1 - \alpha_i) L_{i,t}^{1-\nu} \right)^{\frac{1}{1-\nu}} \tag{6.1}$$

The elasticity of substitution is chosen to be 0.20, 0.40, 0.60 or 0.80. The experiments are in every other way identical to Fisher's, i.e., a total of 3960 (4×990) unique experiments.

However, a minor problem emerges: in 760 experiments it was not possible to ensure uniform wages in every period through the redistribution of labour between the n firms. This is a consequence of an obvious mathematical property of the CES function, when capital and technology are *ex ante* given. To circumvent this problem, all of these experimental sessions, in which it was not possible to determine a set of uniform wage rates in one or more periods, have been removed. Consequently, the following results are based on 3200 ($3960 - 760$) experiments.

Inspecting the correlation coefficients and standard deviations of labour's share from these 3200 unique experiments show that 81 percent of the correlation coefficients are greater than 0.90 and 59 percent are greater than 0.99. In the 595 time series with a non-trend stationary shift term, 80 percent are greater than 0.90 and 44 percent are greater than 0.99. Moreover, there is no clear connection between the standard deviation of labour's share and the correlation coefficients, i.e., again it is shown that under very general circumstances, there is a high risk that this kind of empirical work will result in fundamentally misleading conclusions about the underlying technology.

That *only* 44 percent of the series with a non-trend stationary shift are greater than 0.99 emphasises; that these findings do not generalise Shaikh's result that guaranties a perfect fit under the more restrictive conditions, but simply imply that it is very likely to obtain a very good fit under very general circumstances.

Again the series are checked for potential spurious regressions. The tests show that there is potential risk of spurious regression in 7 percent of the 3200 regressions, i.e., the high correlation coefficients again cannot be explained by spurious regression.

[5] Fisher et. al. (1977) analysed wage explanation in simulations with CES micro production functions. In this study aggregate Cobb-Douglas as well as CES production functions were estimated and in general both types fit the data well, as long as the shares were stable.

To sum up, the results from the extended model also support a more general version of Shaikh's critique, because even though the likelihood of observing near perfect correlation drops, when assumptions A and B are violated, it is still very likely to obtain correlation coefficients that most researchers would (wrongly) interpret as support for the estimated functional form.

7 Concluding Remarks

Fisher's 1971 computer experiment has been reconstructed and his results verified. Strengthened by the extensions we have employed, Shaikh's original findings have been confirmed along with the extensions presented in Dixon and McCombie (1991) and Felipe and Holz (2001). We show that even under the general circumstances where neither Shaikh's Perfect Fit Theorem nor the results presented in Dixon and McCombie (1991) and Felipe and Holz (2001) would predict a (near) perfect fit, the Cobb-Douglas production function still shows an "impressive" ability to mimic the data, even with the most simple and popular econometric method.

The implications of these cumulative results are important because they imply that empirical studies, in which a Cobb-Douglas production function is estimated, are necessarily inconclusive. This undermines empirical support for the neoclassical theory of growth and distribution, because that support — to a wide extent — is based on the Cobb-Douglas production function. Moreover, it is a serious warning against using *AS IF* justifications for economic theory.

The lesson from this exercise should be that extreme caution is necessary when applying aggregate production functions; indeed, instead of aggregate production functions, we would propose the implementation of (physical) multi-sector input–output systems in general macroeconomic models, because there is neither theoretical nor empirical support for the use of aggregate production functions. In our opinion, an aggregate production function is simply a notion used for mathematical convenience and elegance.

Some might argue that a more "realistic" production function like the (nested) CES or translog would overcome these problems, but the Cobb-Douglas function's ability to fit (plausible and implausible) data are of course fully embedded in the more flexible functional forms.

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A The Experiments

Fisher (1971) presents a neoclassical model comprising of n firms each producing one homogeneous output. The inputs consist of homogeneous and perfectly mobile labour and heterogeneous capital and technology that are bound to the individual firms. The experiments run for 20 periods, $t = 1, 2, ..., 20$.

Production at the ith firm is either modelled by a Cobb-Douglas or CES production function, *viz.*

$$y_{i,t} = A_{i,t} L_{i,t}^{\alpha_i} K_{i,t}^{1-\alpha_i} \tag{A.1}$$

$$y_{i,t} = A_{i,t} \Big[\alpha_i L_{i,t}^{1-\nu} + (1-\alpha_i) K_{i,t}^{1-\nu} \Big]^{\frac{1}{1-\nu}} \tag{A.2}$$

Where ν is the reciprocal of the elasticity of substitution between labour and capital. The aggregate production function and the associated aggregate capital stock is given by:

$$Y_t = A_t L_t^{\alpha} J_t^{1-\alpha} \tag{A.3}$$

$$J_t = \sum_{i=1}^{n} \left(\frac{\sum_{t=1}^{20} r_{i,t} K_{i,t}}{\sum_{t=1}^{20} K_{i,t}} \right) K_{i,t} \tag{A.4}$$

Wages and profits are paid their marginal products and it is assumed that labour is distributed such that the wage level coincides across the n firms. The algorithm applied to ensure distribution of labour is presented.

In all experiments, the evolution of the total supply of labour, Hicks neutral technology, and capital endowments are exogenously given by:

$$L_t = \exp(0.3t + 0.02\varepsilon_t) \qquad \varepsilon_t \sim \mathrm{N}(0,1) \tag{A.5}$$

$$A_{i,t} = \exp(\gamma_{i,1} t) \qquad v_{i,t} \sim \mathrm{N}(0,1) \tag{A.6}$$

$$K_{i,t} = \exp(\beta_{i,0} + \beta_{i,1} t + 0.0001\eta_{i,t}) \qquad \eta_{i,t} \sim \mathrm{N}(0,1) \tag{A.7}$$

The experiments include two, four, or eight firms. Depending on this, the parameter, α_i, from the production function can take the following values:

$$n = 2: \quad (\alpha_1, \alpha_2) \in \big\{(0.7, 0.8), (0.6, 0.9)\big\}$$

$$n = 4: \quad (\alpha_1, ..., \alpha_4) \in \big\{(0.6, 0.7, ..., 0.9), (0.7, 0.725, ..., 0.8)\big\}$$

$$n = 8: \quad (\alpha_1, ..., \alpha_8) \in \big\{(0.6, 0.6 + \tfrac{1}{7}0.3, ..., 0.9), (0.7, 0.7 + \tfrac{1}{7}0.1, ..., 0.8)\big\}$$

The initial capital endowments can be distributed in three different ways, *viz.*

1. $\beta_{i,0} = 0 \;\; \forall \;\; i = 1, 2, ..., n$

2. $\beta_{i,0} = 0 \;\; \forall \;\; i = 1, 2, ..., \frac{n}{2}$

$$\beta_{i,0} = 2 \;\; \forall \;\; i = \tfrac{n}{2}, \tfrac{n}{2}+1, ..., n$$

3. $$\begin{aligned} \beta_{i,0} &= 2 \;\; \forall \;\; i = 1, 2, ..., \tfrac{n}{2} \\ \beta_{i,0} &= 0 \;\; \forall \;\; i = \tfrac{n}{2}, \tfrac{n}{2}+1, ..., n \end{aligned}$$

Finally, the experiments fall into the following five groups:

1. Two group capital

$$\begin{aligned} &\beta_{i,1} \in \{-0.05, -0.04, ..., 0.05\} && \forall \;\; i = 1, 2, ..., \tfrac{n}{2} \\ &\beta_{i,1} = 0 && \forall \;\; i = \tfrac{n}{2}, \tfrac{n}{2}+1, ..., n \\ &\gamma_{i,1} = 0 && \forall \;\; i = 1, 2, ..., n \end{aligned}$$

2. Two group Hicks preliminary

$$\begin{aligned} &\gamma_{i,1} \in \{-0.05, -0.04, ..., 0.05\} && \forall \;\; i = 1, 2, ..., \tfrac{n}{2} \\ &\gamma_{i,1} = 0 && \forall \;\; i = \tfrac{n}{2}, \tfrac{n}{2}+1, ..., n \\ &\beta_{i,1} = 0 && \forall \;\; i = 1, 2, ..., n \end{aligned}$$

3. Two group Hicks

$$\begin{aligned} &\gamma_{i,1} \in \{(\alpha_i - 1)0.05, (\alpha_i - 1)0.04, ..., (1 - \alpha_i)0.05\} && \forall \;\; i = 1, 2, ..., \tfrac{n}{2} \\ &\gamma_{i,1} = 0 && \forall \;\; i = \tfrac{n}{2}, \tfrac{n}{2}+1, ..., n \\ &\beta_{i,1} = 0 && \forall \;\; i = 1, 2, ..., n \end{aligned}$$

4. The fanning capital

$$\begin{aligned} &\beta_{i,1} \in \{(i - 1)0.05, (i - 1)0.04, ..., (i - 1)0.05\} && \forall \;\; i = 1, 2, ..., n \\ &\gamma_{i,1} = 0 && \forall \;\; i = 1, 2, ..., n \end{aligned}$$

5. The fanning Hicks

$$\begin{aligned} &\gamma_{i,1} \in \{(\alpha_i - 1)(i - 1)0.05, (\alpha_i - 1)(i - 1)0.04, ..., (1 - \alpha_i)(i - 1)0.05\} \\ & \qquad\qquad\qquad\qquad \forall \;\; i = 1, 2, ..., n \\ &\beta_{i,1} = 0 \qquad\qquad\qquad\quad \forall \;\; i = 1, 2, ..., n \end{aligned}$$

This concludes the description of the experiments which by simple combinatorial calculation amounts to 990 unique settings.

The algorithm, used to distribute labour among the n firms such that the wage levels are approximately equal across the firms, has the following structure.

1. Distribute the initial endowments of capital and technology.
2. Uniformly distribute the total labour supply across the n firms and compute the n wage levels.
3. Allocate a given amount of labour from the firms with a low wage level to the the firms with high wage level.

4. Repeat step three until the maximum deviation among the wage levels are less than 1 percent.

Computation of this, however, is not always straightforward because the production functions satisfy the *Inada Conditions*, i.e., when labour inputs are close to zero small changes have large effect on the marginal products. The solution is to dynamically reduce the allocation of labour as the wage levels converge.