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DYNAMICAL COUPLING, NONLINEAR ACCELERATOR AND THE PERSISTENCE OF BUSINESS CYCLES♣

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♣ I would like to thank my mentor, colleague and friend, Vela Velupillai, for being a constant source of inspiration and for having, many years ago, introduced me to Richard Goodwin. Special thanks also to the Trento members of the *Algorithmic Social Sciences Research Unit* (**ASSRU**), Selda Kao and Ragupathy Venkatachalam, for their constant help and support.

Abstract

Many of the research questions and research programs that were posed and suggested by Richard Goodwin are still highly relevant for various methodological, empirical and theoretical reasons. In this paper, we address the issue of whether highly articulated and complex economic interactions can be represented by a simplified low dimensional model. We follow the valuable insight offered by Goodwin (1947) concerning the importance of dynamical coupling and the potential role of analog and/or digital computers in studying them fruitfully. Here, we extend the nonlinear, *flexible accelerator - dynamic multiplier* model of business cycle by Goodwin (1951), to the case in which these economies are coupled through trade. The dynamics implied by the coupling are studied in analogy with the well known Fermi-Pasta-Ulam problem (Fermi et al., 1955). We show that for nonlinear economies, even when they are exactly the same, i.e. having the same structural behavioral equations, the very rich dynamics depend crucially on the initial conditions. This result is somewhat unexpected.

Keywords: Business cycles, coupled economies, analog and digital computations, Fermi-Pasta-Ulam problem, computable economics, algorithmic social sciences, Richard Goodwin, analogy and induction

1 Introduction

Richard Goodwin has made several important contributions to economics (Velupillai, 1998a,b). Probably, one of his well known contributions is the nonlinear business cycle model - the *dynamic multiplier-flexible accelerator*, Goodwin (1951) model.

The multiplier-accelerator model (Goodwin, 1951) and the growth cycle model (Goodwin, 1967) are highly aggregated models, where the variables are scalar. The issue of whether such a high level of aggregation is justified was well recognized and discussed in several works by Goodwin (1946, 1947, 1949). In these models, there are two types of aggregation problems that emerge. The first one is connected with the aggregation of different physical entities that have different physical dimensions into one scalar (index) and the second is connected with the coupling of different markets which have different *production lags* into one *representative* market. In Goodwin (1947) the problem of studying the collective behavior of n-markets as if it were just one market is fully acknowledged:

> To go from two identical markets to n nonidentical ones will require the prolonged services of yet unborn calculating machines. If aggregation is satisfactory, we may, of course, reduce the larger problem to one involving only a few sectors, ...(Goodwin, 1947, p.204).

In this paper, we show that even in the case of coupled ‘identical economies’[1], described by the same structural and behavioral equations, however, differing only in their initial conditions, the practice of aggregation *may not be satisfactory*. We investigate this problem using the *services of calculating machines* that were unborn in 1947.

2 Goodwin’s flexible accelerator and the business cycle

Goodwin’s article *The Nonlinear Accelerator and the Persistence of Business Cycles*, Goodwin (1951) was published after a series of contributions in which he addressed the issue of oscillatory behavior from various angles. Oscillatory behavior - seen as an irregular development as in *Innovations and the irregularity of economic cycles* Goodwin (1946); due to the strength of the coupling between different sectors as in *Dynamical Coupling with Especial Reference to Markets Having Production Lags*, Goodwin (1947); or by considering the effect of different time reactions in the Keynesian income-expenditure-production multiplier framework in *The Multiplier as*

[1] These identical economies may be viewed as identical sectors.

a Matrix, Goodwin (1949); in the review of Hicks' *A Contribution to the Business Cycles*, (Hicks (1950); Goodwin (1950)).

Goodwin's 1951 model can be viewed as a synthesis of his previous works and it captures the essence of the coordination problem addressed by Harrod, by considering the difference between the investment decisions made by the producers and the consumption decisions made by the consumer. The investment function is what is now known as the accelerator model[2]

Both Hicks (1950) and Goodwin (1951) substantially ameliorated the Frisch (1933), Samuelson (1939), Harrod (1939) linear accelerator theory of investment. In a nutshell, they consider the producer's decisions regarding investment as being guided by the desired level of capital (in the following we will consider only Goodwin's description) and the desired capital is linked to the technical ratio determined by the proportion existing between aggregate capital and output:

$$K^d(t) = v_\sigma Y^e(t) \tag{2.1}$$

The desired level of capital, $K^d(t)$, depends on the expected demand, $Y^e(t)$, where a fixed proportion between quantity of capital necessary for production is, following Harrodian lines, technically determined by the capital-output ratio v_σ. If we assume that the expected sales are equal to current actual sales, $Y^e(t) = Y(t)$ and that the desired capital becomes, after a lag, $\theta + \epsilon$, actual capital, $K^d(t) = K(t)$ we have that under *normal-linear* conditions the investment will be:

$$I(t+\theta+\epsilon) = \dot{K}(t+\theta+\epsilon) = v_\sigma \dot{Y}(t) \tag{2.2}$$

The capital accumulation equation becomes:

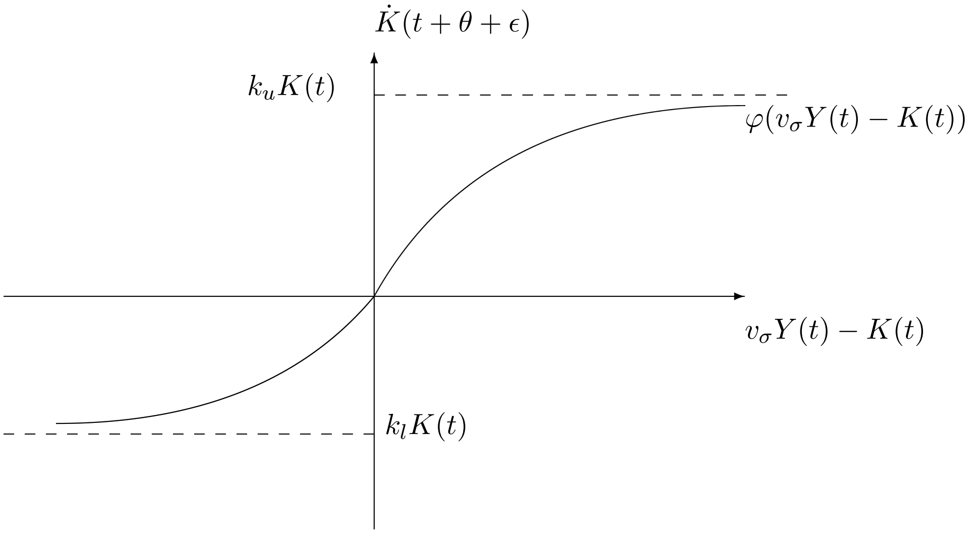
$$K(t+\theta+\epsilon) = K(t) + \int_t^{t+\theta+\epsilon} I(t+\theta+\epsilon)dt \tag{2.3}$$

Hicks and Goodwin explicitly consider the (*nonlinear*) limits set by demand (limits to full employment level of demand – Hicks) and limits set by the impossibility of the immediate adjustment to the desired levels of capital (it takes time to build – Goodwin) [3]. This explicit consideration of nonlinearity was absent in the previous accelerator models by Frisch (1933); Kalecki (1935); Samuelson (1939).

[2]The investment function as *the accelerator* is one of the first theories of investment decisions that had been formalized. See Zambelli (2007) for a summary of the debate on proper causality concerning investment that occurred contemporaneously with the birth of the *econometric society and the beginning of mathematization of economics.* (Clark, 1917, 1931, 1932; Frisch, 1931, 1932b,a, 1933). The accelerator is also an integral part of Harrod (1939) and was made famous also by Samuelson (1939) in the linear business cycle model

[3]The model with both the investment lag and the consumption lag has the structure of the Van der Pol (R and of the existance proofs of the limit cycles see Ragupathy and Velupillai (2012).

Fig. 2.1: *Flexible accelerator (modified)*



The investment function is given by the following relationship

$$I(t+\theta+\epsilon) = \dot{K}(t+\theta+\epsilon) = \varphi(v_\sigma Y(t) - K(t)) \quad (2.4)$$

where: $I(t+\theta+\epsilon)$ and $\dot{K}(t+\theta+\epsilon)$ are investment completed and delivered at time $(t+\theta+\epsilon)$, and $\theta+\epsilon$ is the investment lag; $\varphi(v_\sigma Y(t) - K(t))$ is the investment function. Which has the following properties:

$$\lim_{Y(t)\to\infty} \varphi(v_\sigma Y(t) - K(t)) = k_u K(t);$$

$$\lim_{Y(t)\to-\infty} \varphi(v_\sigma Y(t) - K(t)) = k_l K(t);$$

$$\frac{\delta\varphi(Y(t))}{\delta Y(t)} = v_\sigma \text{ when } v_\sigma Y(t) = K(t)$$

Here $k_u > 0$ and $k_l < 0$.

From Fig: 2.1, we see that when the amount of desired capital, $K^d(t) = v_\sigma Y(t)$ is close to the actual available capital $K(t)$, the investment level is $\dot{K}(t) = v_\sigma \dot{Y}(t)$. But, when the desired capital level is above the actual available capital, the rate of investment is constrained by the productive capacity, which explains the rationale for the presence of a ceiling, $k_u K(t)$, (with $k_u > 0$). When the desired capital is much lower with respect to the actual capital, the rate of *destruction* of capital is limited by the natural wearing of the capital and this, in turn, is a function of the available capital,

$k_l K(t)$, (with $k_l < 0$) [4] [5].

The consumption function here is the traditional Keynesian type,

$$C(t + \theta + \epsilon) = C_0 + cY(t + \theta) \tag{2.5}$$

where $C(t+\theta+\epsilon)$ is consumption at time $t+\theta+\epsilon$, C_0 is the autonomous level of consumption, c is the marginal propensity to consume and $Y(t)$ is income at time t. We assume that the moment in which the income is actually earned and the time in which the consumption expenditure will actually take place are separated by an interval of length ϵ.

The national income identity for the closed economy requires:

$$Y(t + \theta + \epsilon) \equiv C(t + \theta + \epsilon) + I(t + \theta + \epsilon) \tag{2.6}$$

Substituting 2.4 and 2.5 into 2.6 we obtain:

$$Y(t + \theta + \epsilon) \equiv C_0 + cY(t + \theta) + \varphi(v_\sigma Y(t) - K(t)) \tag{2.7}$$

Following Goodwin (1951), we approximate $Y(t + \theta + \epsilon)$ and $Y(t + \theta)$ with their respective Taylor series expansions and we truncate to the first

[4] Figure A.1, p.30, reports the function used by Goodwin, which is simpler because the upper and lower bounds are constant and because the domain is $\dot{Y}(t)$ and not $v_\sigma Y(t) - K(t)$, which is the case in our specification. We think that the above formulation is, although minor, an improvement to the original formulation, see Zambelli (2011b).

[5] (See also below, footnote 17, p.24) Goodwin had been a pupil of Schumpeter and completely endorsed his view that business cycles were characterized, among other things, by an asymmetry in the oscillation, i.e. different dynamics during upturn and the downturn. In his famous article, *Propagation Problems and Impulse Problems*, Frisch reports Schumpeter view:

> Suppose that we have a pendulum freely suspended to a pivot. Above the pendulum is fixed a receptacle where there is water. A small pipe descends all along the pendulum, and at the pendulum the pipe opens with a valve which has a peculiar way of functioning. The opening of the valve points towards the left and is larger when the pendulum moves towards the right than when it moves towards the left. Concretely one may, for example, assume that the valve is influenced by the air resistance or by some other factor that determines the opening of the valve as a function of the velocity of the pendulum. Finally we assume that the water in the receptacle is fed from a constantly running stream which is given as a function of time. The stream may, for instance, be a constant (Frisch, 1933, pp.203-204).

We think that Goodwin captured this requirement by Schumpeter through his flexible accelerator model. In addition, the functional specification of the accelerator $\varphi(v_\sigma Y(t) - K(t))$ presented here may be an improvement to Goodwin's original formulation for the following reason: the speed with which 'the valve opens is a function of the upper and lower boundaries ($k_u K(t)$ and $k_l K(t)$). They both, in turn, are a function of the available capital, which is a function of the flow of investment. The latter, in turn, is a function of consumption and foreign demand and so on. Goodwin deserves the credit for having pointed this out to Schumpeter and for fully realizing the importance of nonlinearity in order to understand economic processes.

element. Hence:

$$Y(t+\theta+\epsilon) = Y(t+\theta) + \epsilon\dot{Y}(t+\theta) + \dots \tag{2.8}$$

We further approximate this by using the Taylor series expansion and truncation of $Y(t+\theta)$ and $\dot{Y}(t+\theta)$ respectively and we obtain:

$$Y(t+\theta) = Y(t) + \theta\dot{Y}(t) + \dots \tag{2.9}$$

$$\dot{Y}(t+\theta) = \dot{Y}(t) + \theta\ddot{Y}(t) + \dots \tag{2.10}$$

Substituting equations 2.9, 2.10 into 2.8 and in turn substituting into 2.7 we obtain:

$$Y(t)+(\theta+\epsilon)\dot{Y}(t)+\theta\epsilon\ddot{Y}(t) = C_0+c(Y(t)+\theta\dot{Y}(t))+\varphi(v_\sigma Y(t)-K(t)) \tag{2.11}$$

$$\ddot{Y}(t) = \frac{1}{\theta\epsilon}\Big(C_0+(1-c)Y(t)+((c-1)\theta+\epsilon)\dot{Y}(t))+\varphi(v_\sigma Y(t)-K(t))\Big) \tag{2.12}$$

We define

$$\dot{Y}(t) = Z(t) \tag{2.13}$$

and substitute it into 2.12 to obtain the following nonlinear differential equation:

$$\dot{Z}(t) = \frac{1}{\theta\epsilon}\Big(C_0+(1-c)Y(t)+((c-1)\theta+\epsilon)Z(t))+\varphi(v_\sigma Y(t)-K(t))\Big) \tag{2.14}$$

The equations 2.13, 2.14 and 2.4 define a system of nonlinear differential equations [6]. Given a functional form satisfying the requirements of φ described in Figure 2.1 (see the Appendix A, page 30) and a set of initial conditions, the above system can be numerically integrated. The above planar system may have a point attractor or a limit cycle.

3 Dynamical coupling

A version of the above set of relations describing the closed system has been analyzed, Zambelli (2011b), for the case where there are n economies coupled through trade[7]. The National Accounts Identities are given by:

$$\mathbf{M}(t+\boldsymbol{\theta}+\boldsymbol{\epsilon})+\mathbf{Y}(t+\boldsymbol{\theta}+\boldsymbol{\epsilon}) \equiv \mathbf{C}(t+\boldsymbol{\theta}+\boldsymbol{\epsilon})+\mathbf{I}(t+\boldsymbol{\theta}+\boldsymbol{\epsilon})+\mathbf{X}(t+\boldsymbol{\theta}+\boldsymbol{\epsilon}) \tag{3.1}$$

[6] Actually the system is composed of two differential equations, 2.13 and 2.14 and one difference-differential equation, 2.4. In the context of this paper, it is not important to stress this distinction and hence we will consider the above system as one that is composed of differential equations. In the numerical simulations presented below, this difference has in its substance been kept. This means that the initial conditions will not have to be the three point values $Z(0), Y(0), K(0)$, but will have to be the functions defined in the interval $[t_0-(\theta+\epsilon), t_0]$

[7] Here, we will simplify this compared to the formulation in Zambelli (2011b) because, here, we will not be considering the eventual (small) effects on the countries foreign expenditures determined by the differences in the countries different consumption and investment lags

Here: $\mathbf{M}(t+\boldsymbol{\theta}+\boldsymbol{\epsilon})$, is the vector of imports; $\mathbf{Y}(t+\boldsymbol{\theta}+\boldsymbol{\epsilon})$ is the vector of domestic national incomes; $\mathbf{C}(t+\boldsymbol{\theta}+\boldsymbol{\epsilon})$ is the vector of national consumption levels; $\mathbf{I}(t+\boldsymbol{\theta}+\boldsymbol{\epsilon})$ is the vector of investment levels; $\mathbf{X}(t+\boldsymbol{\theta}+\boldsymbol{\epsilon})$ is the vector of exports; The consumption lag relative to each country is given by the vector $\boldsymbol{\epsilon}$ and the investment lag by the vector $(\boldsymbol{\theta}+\boldsymbol{\epsilon})$. Clearly, the element i of each vector is to be associated to the i^{th} country.

Imports of the countries are determined by a linear function of its income (with lag $\boldsymbol{\epsilon}$):

$$\mathbf{M}(t+\boldsymbol{\theta}+\boldsymbol{\epsilon}) = [diag(\mathbf{m})]\mathbf{Y}(t+\boldsymbol{\theta}) \tag{3.2}$$

where $\mathbf{m}$ is the vector of the marginal propensities to import [8]. Resorting to Taylor series expansions and applying the same procedure as for the single closed economy, we have:

$$\mathbf{M}(t+\boldsymbol{\theta}+\boldsymbol{\epsilon}) = [diag(\mathbf{m})][\mathbf{Y}(t) + [diag(\boldsymbol{\theta})]\dot{\mathbf{Y}}(t)] \tag{3.3}$$

The consumption levels of countries' are determined by:

$$\mathbf{C}(t+\boldsymbol{\theta}+\boldsymbol{\epsilon}) = \mathbf{C}_0 + [diag(\mathbf{c})]\mathbf{Y}(t+\boldsymbol{\theta}) \tag{3.4}$$

where $\mathbf{c}$ is the vector of marginal propensities to consume.

Through Taylor series expansion, we obtain:

$$\mathbf{C}(t+\boldsymbol{\theta}+\boldsymbol{\epsilon}) = \mathbf{C}_0 + [diag(\mathbf{c})][\mathbf{Y}(t) + diag(\boldsymbol{\theta})\dot{\mathbf{Y}}(t)] \tag{3.5}$$

The vector of exports, $\mathbf{X}(t+\boldsymbol{\theta}+\boldsymbol{\epsilon})$ is determined by the decisions of imports of "coupled" countries:

$$\mathbf{X}(t+\boldsymbol{\theta}+\boldsymbol{\epsilon}) = \boldsymbol{\Lambda}\mathbf{M}(t+\boldsymbol{\theta}+\boldsymbol{\epsilon}) \tag{3.6}$$

where $\boldsymbol{\Lambda}$ is a semi-positive nxn matrix where the major diagonal is zero. The sum of the elements of columns of $\boldsymbol{\Lambda}$ is equal to 1. This is so because the elements in column j are the imports split by origin, so that the total amount imported must be equal to 1. The maximum value of the sum by rows of $\boldsymbol{\Lambda}$ can be at most $(n-1)$, and this is the case in which there is only one country which is an exporter.

Substituting equation 3.2 into 3.7 we obtain:

$$\mathbf{X}(t+\boldsymbol{\theta}+\boldsymbol{\epsilon}) = \boldsymbol{\Lambda}[diag(\mathbf{m})][\mathbf{Y}(t) + [diag(\boldsymbol{\theta})]\dot{\mathbf{Y}}(t)] \tag{3.7}$$

Following the same approach as for the single economy, we can truncate the Taylor series expansion of

$$\mathbf{Y}(t+\boldsymbol{\theta}+\boldsymbol{\epsilon}) = \mathbf{Y}(t) + ([diag(\boldsymbol{\theta}] + [diag(\boldsymbol{\epsilon})])\dot{\mathbf{Y}}(t) + [diag(\boldsymbol{\theta})][diag(\boldsymbol{\epsilon})]\ddot{\mathbf{Y}}(t) \tag{3.8}$$

[8]Given a vector $[diag(\cdot)]$ is a diagonal matrix having the elements in the main diagonal equal to the element of the vector and the remaining elements equal to zero

We are now left to determine the dynamic evolution of capital. Investment will be determined by an extension to the n^{th} case of eq. 2.4

$$\dot{\mathbf{K}}(t+\boldsymbol{\theta}+\boldsymbol{\epsilon}) = \varphi(\mathbf{v}_\sigma \mathbf{Y}(t) - \mathbf{K}(t)) \tag{3.9}$$

Introducing the transformation $\dot{\mathbf{Y}}(t) = \mathbf{Z}(t)$ and substituting equations 3.3, 3.8, 3.4, 3.2, 3.7 into 3.1 we obtain the following system capturing the dynamics of output:

$$\dot{\mathbf{Z}}(t) = \frac{\mathbf{C}_0 - \boldsymbol{\alpha}_1 \mathbf{Y}(t) - \boldsymbol{\alpha}_2 \mathbf{Z}(t) + \boldsymbol{\varphi}(\mathbf{v}_\sigma \mathbf{Y}(t) - \mathbf{K}(t))}{[diag(\boldsymbol{\theta})][diag(\boldsymbol{\epsilon})]} \tag{3.10}$$

$$\dot{\mathbf{Y}}(t) = \mathbf{Z}(t) \tag{3.11}$$

where:
$\boldsymbol{\alpha}_1 = (\mathbf{I} - [diag(\mathbf{c})]) + (\mathbf{I} - \boldsymbol{\Lambda})[diag(\mathbf{m})]$
$\boldsymbol{\alpha}_2 = ((\mathbf{I} - [diag(\mathbf{c})]) + (\mathbf{I} - \boldsymbol{\Lambda})[diag(\mathbf{m})])[diag(\boldsymbol{\theta})] + [diag(\boldsymbol{\epsilon})]$

The above equations 3.9–3.11 represent a system of $3n$ differential equations of the first order. When n=1 (and hence there are no exports or imports) we have the simple country case. The above system can be integrated numerically by using standard numerical approximations.

The equilibrium of the above system is atttained when $\dot{\mathbf{Y}}(t)$, $\dot{\mathbf{K}}(t)$, $\dot{\mathbf{Z}}(t)$, are all equal to 0. Setting these conditions, we see that the auxiliary $\mathbf{Z}(t)$ has the equilibrium value $\mathbf{Z}^{eq} = 0$. From equation 3.10 we compute the equilibrium value for domestic output:

$$\mathbf{Y}^{eq} = \frac{1}{\boldsymbol{\alpha}_1}\mathbf{C}_0 = \frac{1}{(\mathbf{I} - [diag(\mathbf{c})]) + (\mathbf{I} - \boldsymbol{\Lambda})[diag(\mathbf{m})]}\mathbf{C}_0 \tag{3.12}$$

The above is the typical Hansen-Keynes equilibrium where $\frac{1}{\boldsymbol{\alpha}_1}$ is the vector of Keynesian multipliers extended to the case of international trade.

Consequently the equilibrium vector of capital levels is given by:

$$\mathbf{K}^{eq} = [diag(\mathbf{v}_\sigma)]\mathbf{Y}^{eq} \tag{3.13}$$

A nonlinear dynamical system like the above may converge to point attractors, limit cycles or chaotic attractors. A combination of a low capital-output ratio, low marginal propensities to consume and import, low levels consumption and investment lags would make the equilibrium vector a point attractor. For other set of parameters, one could have the existence of limit cycles and chaotic attractors.

4 The *Fermi-Pasta-Ulam* problem and coupled oscillators

Studying coupled systems for the case of linear model economies is almost trivial because in the case of linear systems. In this case, the individual economies, when they are perturbed away from the equilibrium position, will converge to or diverge away from the equilibrium point. In other words, the system will converge towards the same attractor, independently of the initial conditions and independently of the strength of the coupling. In our case, it would occur independently of the intensity of the trade linkages.

Here, we will try to shed light on the working of the dynamical system 3.9–3.11 when the attractor is a limit cycle, following the method of investigation used by Fermi–Pasta–Ulam (FPU) (Fermi et al., 1955). The FPU-problem is concerned with the study of the distribution of energy between identical units ('atoms') connected in a string - a sort of unidimensional lattice, with the aid of computer simulations. Prior to the experiment, Fermi, Pasta and Ulam were expecting that when excited, i.e. when perturbed from the original equilibrium position, the energy would be distributed evenly to all the *energy modes*. To their surprise, this 'thermalization did not happen and the energy was not dispersed uniformly. This experiment led to a stream of research, which is still taking place and has, for example, led to the discovery of solitons (Weissert, 1997) Galavotti (2008)

The FPU problem and the way computer simulations were effectively utilized to understand a theoretical problem is instructive for economists, in particular, for the tailoring laboratory experiments. It is particularly relevant here to quote Ulam:

> Fermi became interested in the development and the potentialities of electronic computing machines ... We decided to try a selection of problems for heuristic work where in absence of closed analytical solutions experimental work on computing machines would perhaps contribute to the understanding of properties of solutions. This could be particularly fruitful for problems involving the asymptotic–long time or "in the large" behavior of non-linear physical system. *In addition, such experiments on computing machines would have at least the virtue of having the postulates clearly stated. This is not always the case in an actual physical object or model where the assumptions are not perhaps explicitly recognized*(Fermi et al., 1955, p.977, emphasis added)[9]

[9] Vela Velupillai and Ragupathy Venkatachalam have both pointed out to me the similitude between the FPU problem and the work on morphogenesis by Turing (1952). In particular in FPU and in Turing's paper the placing of the single atoms or molecules on the ring is very similar. (Velupillai, 2003, pp.36-41) has made a strong link between

In analogy with the FPU case, the countries are located on a unidimensional lattice (actually a ring) where they trade only with their neighbors (left and right)[10]. The n economies are assumed to be identical. This means that they are described by the same structural parameters: the fundamentals of these economies are the same. The choice of the parameters (refer Appendix, B, p. 31) are such that these economies all have the same qualitative behavior as in the case of Goodwin (1951), i.e. the attractor is a limit cycle. Given that the economies are identical, if they are provided with the same initial conditions, they will be fully synchronized. This means that they will be indistinguishable. Now we pose the main question that is of interest to us.

> **Research Question.** Identical individual economies of the type described by the system, eq. 3.9–3.11, whether, independent of their initial conditions, these economies eventually, after a transient period, exhibit a perfectly synchronous behavior?

5 Ulam type *Laboratory Experiment*

We attempt to answer the above question using numerical (computational) laboratory experiments. The parameter values for these economies are given in Appendix B. As mentioned above the export parameters matrix is such that the economies are locally connected (left-right) through trade. That is:

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complexity, economics and Turing bifurcations. Also particularly striking is the reference to the need of the *services of digital computers* made by Turing. In the final section of his paper, *Nonlinear Theory. The use of digital computers*, Turing writes

> " . . . One would like to be able to follow this more general process mathematically also. The difficulties are, however, such that one cannot hope to have any very embracing theory of such processes, beyond the statement of the equations. *It might be possible, however, to treat a few particular cases in detail with the aid of a digital computer. This method has the advantage that it is not so necessary to make simplifying assumptions as it is when doing a more theoretical type of analysis.* . . . Even with the ring problem . . . for which a reasonably complete mathematical analysis was possible, the computational treatment of a particular case was most illuminating (Turing, 1952, p.72, emphasis added).

The FPU approach and the Turing approach appear to be very similar, particularly in the case of the ring. Turing was more concerned with the change in structure (i.e. *morphogenesis*) while FPU were more concerned with why a certain structure would *survive* (i.e. the lack of *thermalization*), but the setting in which they developed their models is almost the same.

[10]The location on the unidimensional lattice (a ring) is here not an essential property of the model. This choice is made for expository reason as well as to make the problem as clear and simple as possible. As it will be seen below and in the Appendix, B, p.31, the location on the ring is equivalent to a particular structure of the export matrix $\mathbf{\Lambda}$

$$\mathbf{\Lambda} = \begin{bmatrix} 0 & \mathbf{0.5} & 0 & \dots & \dots & \dots & \mathbf{0.5} \\ \mathbf{0.5} & 0 & \mathbf{0.5} & 0 & \dots & \dots & 0 \\ 0 & \mathbf{0.5} & 0 & \mathbf{0.5} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \dots \\ \vdots & \vdots & \vdots & \mathbf{0.5} & 0 & \mathbf{0.5} & 0 \\ \vdots & \vdots & \vdots & \vdots & \mathbf{0.5} & 0 & \mathbf{0.5} \\ \mathbf{0.5} & 0 & 0 & \dots & 0 & \mathbf{0.5} & 0 \end{bmatrix} \tag{5.1}$$

We have chosen parameter values so that the economies are identical and converge towards a limit cycle. We begin our experiment by assuming that the $n = 32$ economies are located at the (unstable) equilibrium[11]. At time t_0 two economies (number 16 and number 17) are removed away from equilibrium with a positive 10% shock (see Appendix B, p.31). After a transient period one would expect that the shock would fade away in the sense that it should be 'distributed or transmitted' among the different economic systems. An increase in the internal demand of a country, due to an increase of the domestic output, would determine an increase in the demand of foreign goods from the neighboring countries. This, in turn, would increase the demand of their neighboring countries. Meanwhile, the usual effects on consumption and on investment would take place.

Quite surprisingly, this is not what happens and the shocks do not fade away. Figures 5.1, 5.2 report the results of this experiment: we can see that the economies are a-synchronous. An inspection of 5.2 shows that during the cycle, while some economies are experiencing a boom, others are experiencing a depression. The thick line shows the global average which is 'internal' with respect to the individual countries, simply because the contribution of expanding economies is counterbalanced by the contributions made by those which are contracting. *If the original shock had been absorbed, we should have observed a perfect overlapping of these curves, i.e. we should have observed just one thick line.* Furthermore, the existence of different "circles" shows a difference in the performances of the economic systems. For example, some countries would reach higher levels of income than others and their current accounts would be different as well.

Figures 5.6 show the individual output levels during the reference cycle or limit cycle distributed along the countries' locations (the lattice or the ring). There are 20 snapshots going from the beginning of the reference cycle $t_{cycle} = 1/20$ to the end or beginning of a new round.

[11]Other simulations have been done taking as initial conditions not displacements away from the point equilibrium, but with displacements away from the totally synchronized limit cycles. At time t_0 the displacement is made with the same percent displacements as those out of equilibrium point. The behavior is qualitatively similar in both cases. For reasons of space, these evolutions are not reported here

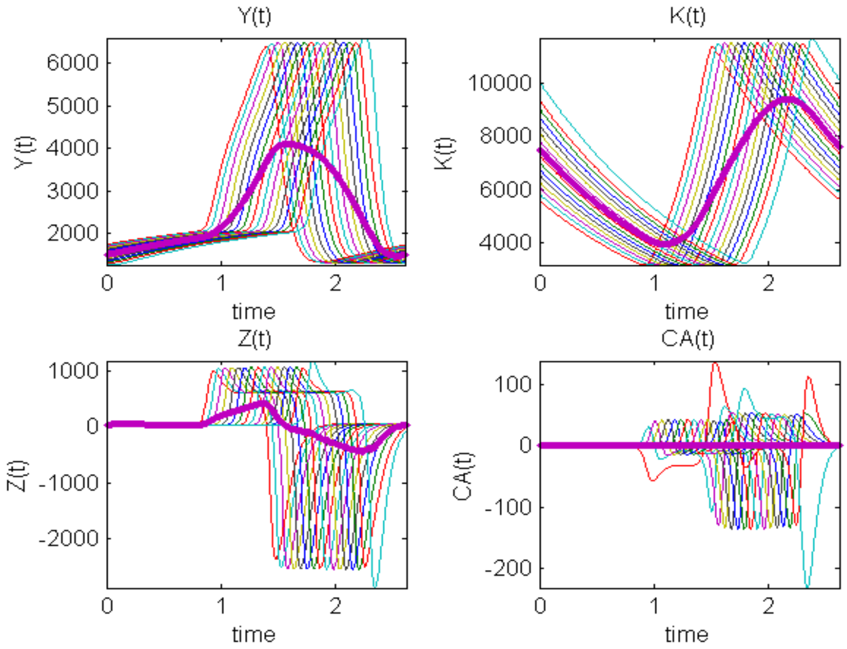

Fig. 5.1: **Single Shock** - Limit Cycle - Time. *Initial conditions at* t_0 *are the equilibrium values, but with* $Y_{16}(t_0) = 1.1Y_{16}^{eq}$ *and* $Y_{17}(t_0) = 1.1Y_{17}^{eq}$ *(see Appendix B). The thick lines are the averages.*

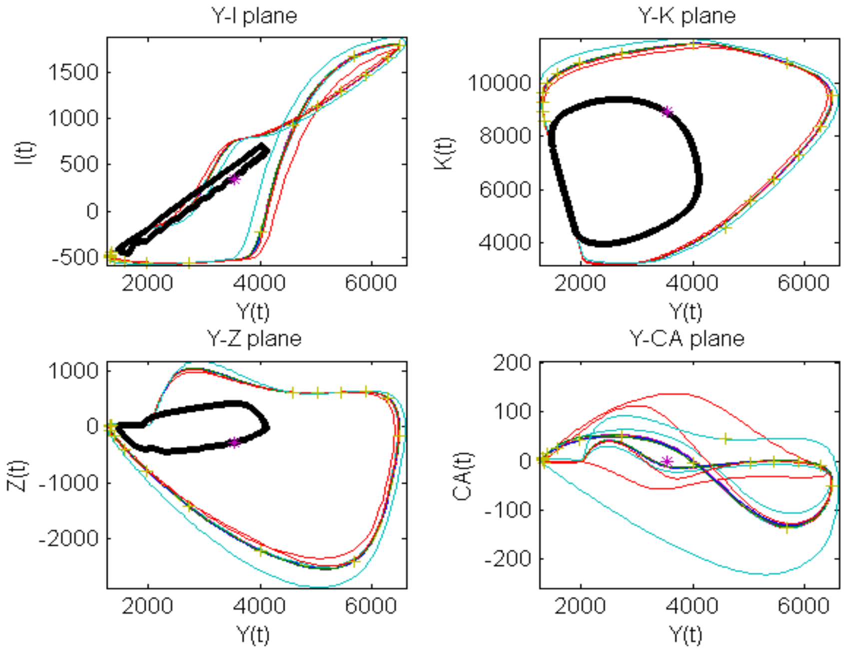

Fig. 5.2: **Single Shock** - Limit Cycle - Plane projections *Initial conditions at* t_0 *are the equilibrium values , but with* $Y_{16}(t_0) = 1.1Y_{16}^{eq}$ *and* $Y_{16}(t_0) = 1.1Y_{16}^{eq}$ *(see Appendix B). The thick lines are the averages.*

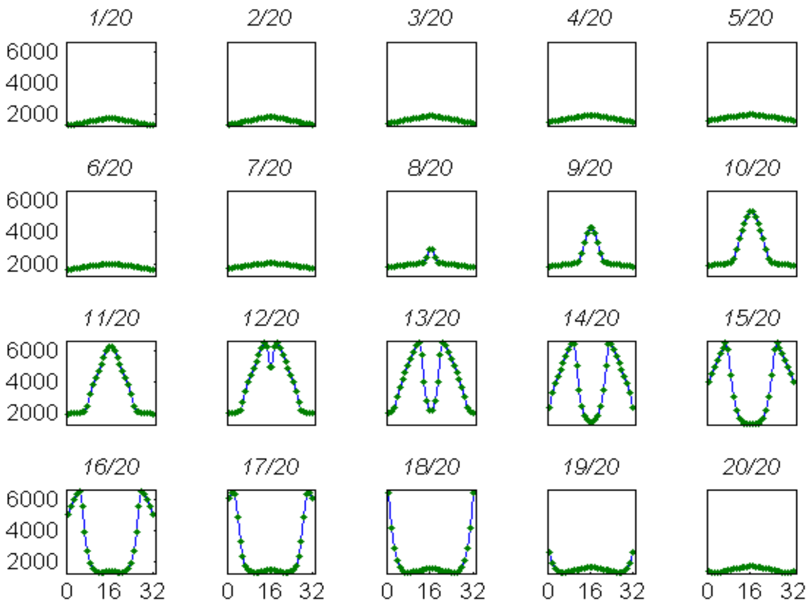

Fig. 5.3: **Single Shock** - Limit Cycle -$\mathbf{Y}(t_{cycle})$. *Unidimensional Lattice. Limit Cycle.* $t_{cycle} = \frac{1}{20} \ldots, \frac{20}{20}$. *Initial conditions: equilibrium and one single shock. See Appendix B*

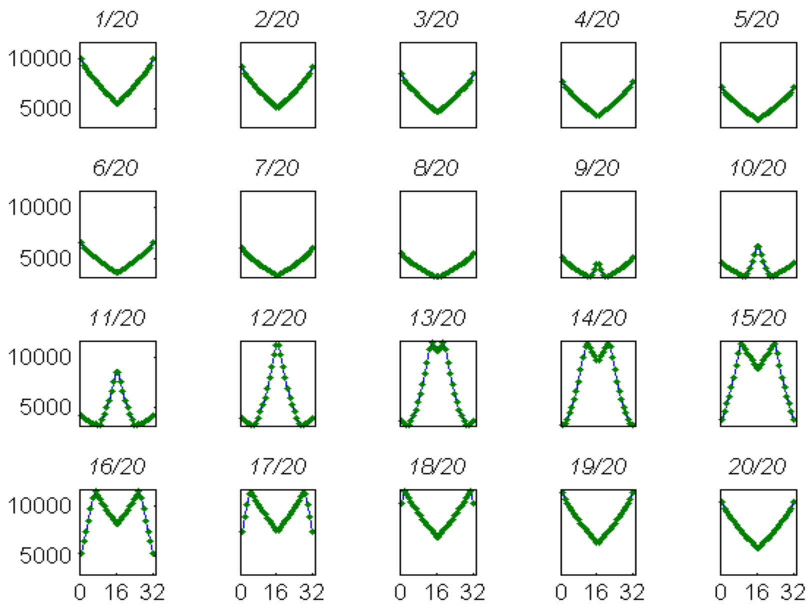

Fig. 5.4: **Single Shock** - Limit Cycle -$\mathbf{K}(t_{cycle})$. *Unidimensional Lattice. Limit Cycle.* $t_{cycle} = \frac{1}{20} \ldots, \frac{20}{20}$. *Initial conditions: equilibrium and one single shock. See Appendix B*

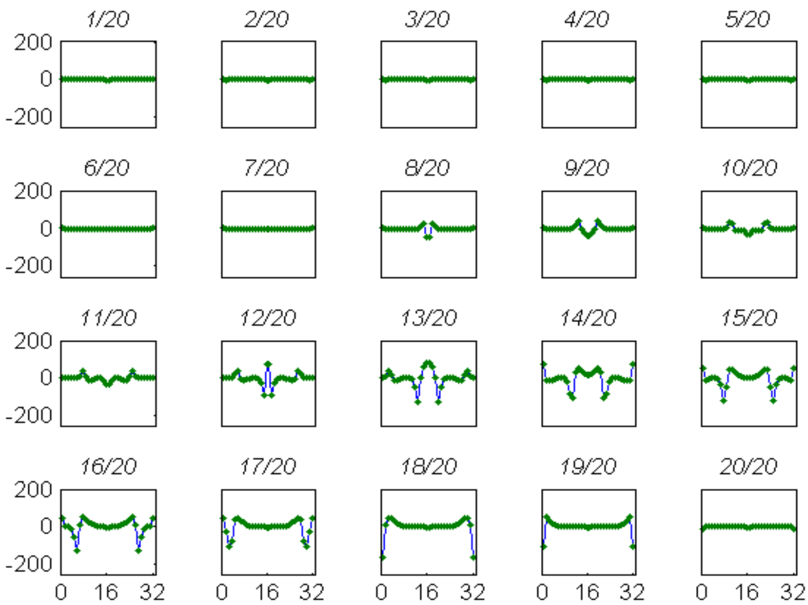

Fig. 5.5: **Single Shock** - Limit Cycle -$\mathbf{K}(t_{cycle})$. *Unidimensional Lattice. Limit Cycle.* $t_{cycle} = \frac{1}{20} \ldots, \frac{20}{20}$. *Initial conditions: equilibrium and one single shock. See Appendix B*

During the first phase of the cycle ($[\frac{0}{20} \ldots \frac{7}{20}]$), all the economies are synchronized at a low level. This is a typical phase of the cycle where production and expenditure are low, investment is negative and exports and imports are low. The presence of excess capacity is obvious from the figure reporting the capital levels (Fig.5.4). The equilibrium of the current account is practically realized for all the economies (Fig.5.5).

The second phase of the cycle ($[\frac{8}{20} \ldots \frac{11}{20}]$) is characterized by a recovery of the central countries. Countries 16 and 17 leave the depression period, due to an increase of investment which is to be explained by an increase of the desired capital. Meanwhile, an increase in exports, which is shown by a deterioration of the current accounts of the central countries Fig.5.5.

The third phase of the cycle ($[\frac{12}{20} \ldots \frac{16}{20}]$)is characterized by the contraction of the central countries and a concurrent expansion of the remaining ones. Here a-synchronicity is high.

The fourth phase of the cycle ($[\frac{12}{20} \ldots \frac{16}{20}]$)is characterized by the depression of the central countries and a contraction of the remaining ones.

One has to keep in mind that these economies have exactly the same structural parameters - but some of them would appear as being leaders and others as being followers. Obviously, *this has nothing to do with their so-called fundamentals, but is only due to the specific initial conditions.*

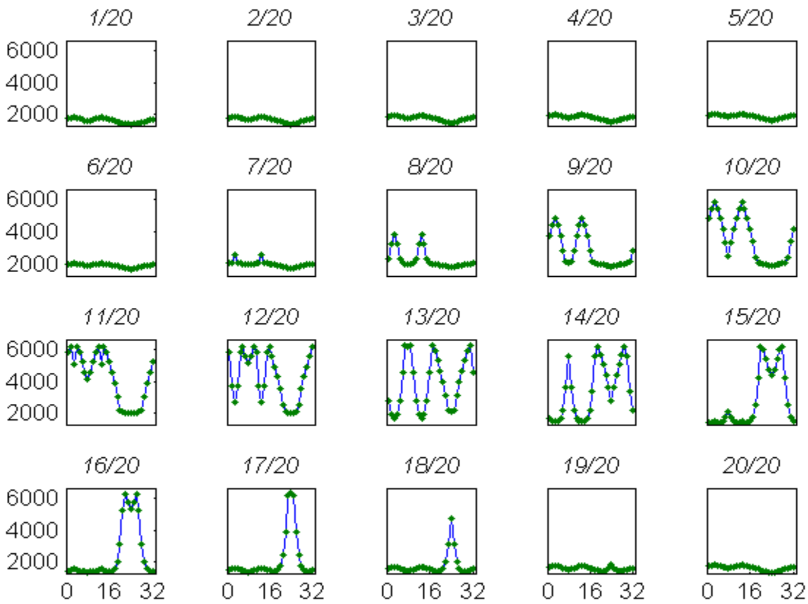

Fig. 5.6: **Sawtooth** - Limit Cycle -$\mathbf{Y}(t_{cycle})$. *Unidimensional Lattice. Limit Cycle.* $t_{cycle} = \frac{1}{20} \ldots, \frac{20}{20}$. *Initial conditions: see Appendix B*

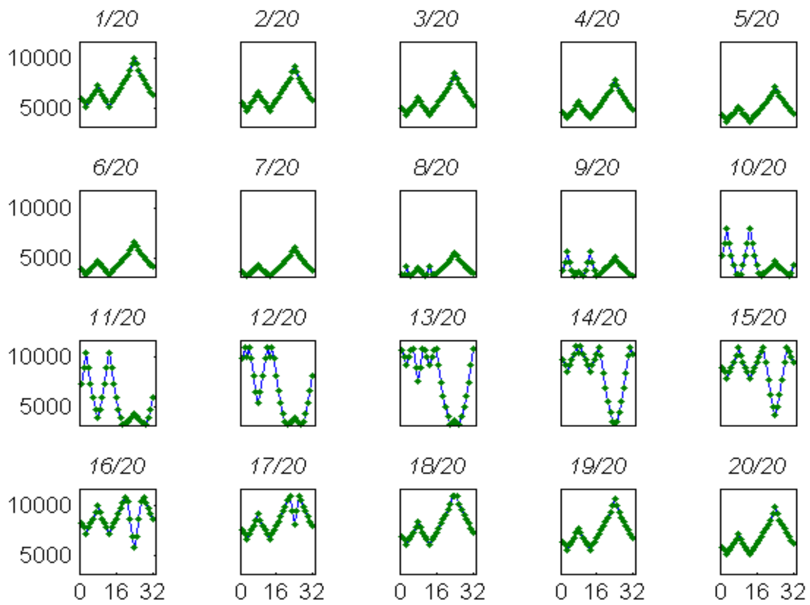

Fig. 5.7: **Sawtooth** – Limit Cycle -**K**(t_{cycle}). *Unidimensional Lattice. Limit Cycle.* $t_{cycle} = \frac{1}{20} \ldots, \frac{20}{20}$. *Initial conditions: see Appendix B*

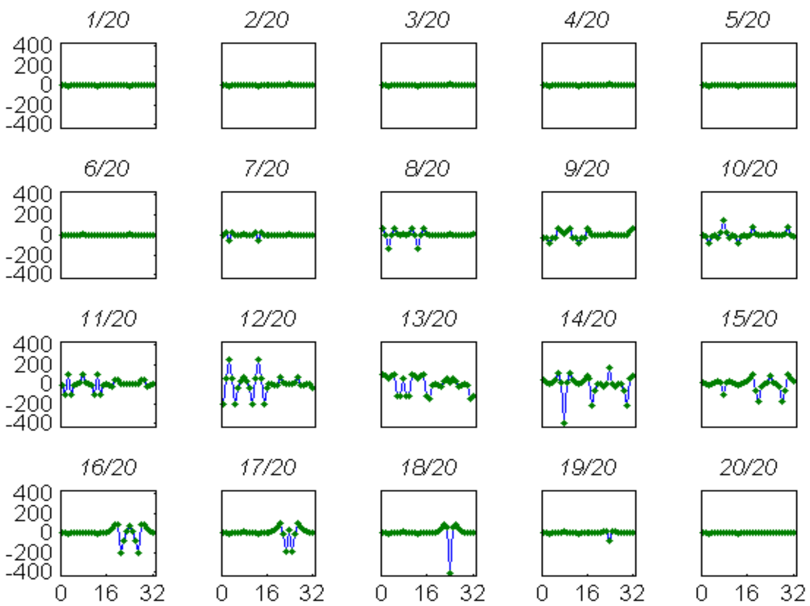

Fig. 5.8: **Sawtooth** - Limit Cycle – **CA**(t_{cycle}). *Unidimensional Lattice. Limit Cycle.* $t_{cycle} = \frac{1}{20} \ldots, \frac{20}{20}$. *Initial conditions: see Appendix B*

Note that the initial conditions are pair-wise equal and this explains the pairwise symmetry which is observed also during the whole reference cycle.

Changing the initial conditions will determine the different, relative dynamic evolutions and the aggregate behavior also would be different. For example, the sawtooth initial conditions reported in the Appendix B determine a different reference cycle compared to the one associated with the case of a single shock.

We can see from Fig. 5.6, a similar long period of depression prevails, but the recovery has a different evolution. Compared to the case of a single shock, there are two countries here, which are not close neighbors which begin the recovery. The countries that are in between them are pulled up (or pulled down) due to the increase in trade activity (see Fig. 5.8). As for capital, its abundance or scarcity indicates whether there is an excess or lack of productive capacity. Note that in this case, the countries located on the left of the lattice are those *'leading'* the system out of the depression. This is opposite to what could have been expected because of the initial conditions, which enabled the right countries with a higher shocks, with respect to the left countries (see Appendix B, Fig. B.2).

Another way to verify that the long run behavior of the system is dependent on the initial conditions is to perform a Fourier transformation and to check whether the averages of the Fourier modes converge or not to the same levels[12]. If the displacement from equilibrium is dispersed we should expect that the averages of the modes converge to the same levels.

We have found, in all our experiments, that this convergence does not take place and this is a further demonstration of the importance of initial conditions. Figure 5.9 shows the case of the individual shock and 5.10 the case when the original shock is the sawtooth. The difference in the shape of the two curves is evident. Furthermore, the sawtooth structure which implies high frequency and alternating behaviors is evident from the alternating values of the mods.

6 The role of expectations

Eq. 2.1 contains expectations and is consistent with the view that the desired productive capacity, i.e. the quantity of capital, is determined by the expected future sales, i.e. expected future demand(Goodwin, 1951, p.4).

[12]We compute the standard normal Fourier modes with the following formula (p.980(reprints p.493) Fermi et al., 1955; Weissert, 1997, p.19)

$$a_k(t) = \sqrt{\frac{2}{n+1}} \sum_{j=1}^{n} (y_j(t) - y_j^{eq}) \sin\left(\frac{jk\pi}{n+1}\right), k = 1, \dots, n$$

where $y_j(t)$ is the output of country j and y^{eq} is the equilibrium value of country j. In this case the equilibrium is the same for all countries: $y^{eq} = y_1^{eq} = \cdots = y_j^{eq} = \cdots = y_n^{eq}$

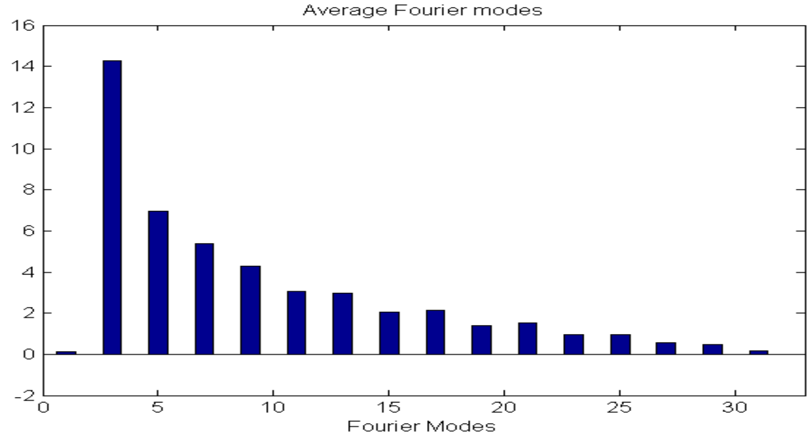

Fig. 5.9: **Shock - Fourier Modes**. *Averages of the Fourier modes. Initial conditions: see Appendix B.*

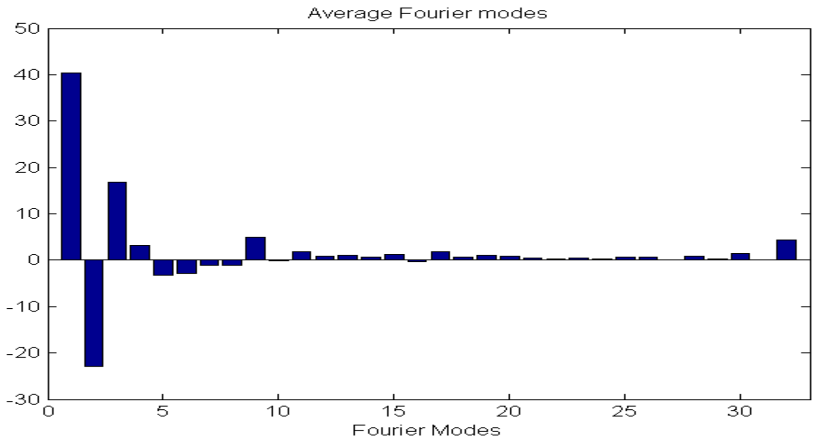

Fig. 5.10: **Sawtooth - Fourier Modes**. *Averages of the Fourier modes. Initial conditions: see Appendix B.*

Also with respect to expectations Goodwin is a precursor In his *coupling* paper Goodwin devotes an important section on the *consequences of expectations*, where he defines a simple aggregate expectation function as the 'average' of the expectations of the participants to the market[13]

> Expectations present a difficult task for economic analysis because there are so many possibilities and little information about their actual behavior. Both expediency and realism suggest the investigation of a very simple type (Goodwin, 1947, p.191)

After having formulated the expectation equation $p^e(t) = p(t-\phi) + \rho\Delta p(t-\phi)$[14], Goodwin (1947) writes:

> It is in contradiction to known facts, and inherently quite unlikely that all producers would have the same expectations . . . The constant ρ will be understood to mean average value of actual individual expectations. Thus its value represents *more group difference of opinion rather than uncertainty* (Goodwin, 1947, p.192 and 193, emphasis added)

All those that have known Goodwin personally will confirm that Goodwin never adhered to the *rational expectation* methodology and to the requirement of *neoclassical* microfoundations. His constructive method and his style refrained him from putting these critical remarks on the works of others in writing. His point of view, maintained up to his last days, is lucidly reported by Herbert Simon in 1997:

> Professor Goodwin reminded us that if we are to arrive at a picture of the economy that can be useful in talking about economic policy or even in describing the operation of that larger economy, we have to solve problems of aggregation . . . All sorts of behavior can take place in non–linear equation systems that simply do not occur in linear systems – qualitatively different behaviors. *If we are to understand dynamic phenomena like the business cycle, it is not enough to understand individual reasoning or how individuals form expectations; it is necessary to put those mechanisms together and to see what predictions they lead to under the difficult circumstances of nonlinearity* (Simon, 1997, p.174, emphasis added)

[13]The work by Goodwin is quoted also in Muth (1961), as an example of the use of expectations for the study of the cob-web dynamics.

[14]Hence in analogy with Goodwin's formulation here we would have: $Y^e(t+\theta+\epsilon) = Y(t) + \rho\Delta Y(t)$

And *those mechanisms* cannot be the unrealistic, uncomputable, undecidable, non–constructive and inhuman rational expectations mechanisms[15].

The scope of the present exercise has been to study the effects of a change in initial conditions on the operation of the whole system. Different aggregate expectations of the Goodwin type would have generated different dynamics, but this would have been the case also with the introduction of different country specific parameters[16]. The aggregate expectation function has been chosen to be the simplest one: the one where, with the proper change of variable ρ is equal to zero (see footnote 14, p.22). This does not at all mean that agents are *naïve*, but only that agents are different and we assume, for simplicity, that the aggregate behavior is such that different groups' opinions even out.

7 Conclusion

In this paper, we have presented the computer laboratory experiments, where identical model economies of the Goodwin multiplier-accelerator type were coupled through trade.

The attractors of these economies may be point attractors or limit cycles. We have shown that the limit cycles of the individual coupled economies and the limit cycle of the aggregate system depend on the initial conditions. In particular, the behavior of each limit cycle may be highly asynchronous. *This was unexpected when we set out to explore the problem.*

If the initial conditions are responsible for the degree and for the type of asynchronicity, it is clear that the eventual policy conclusions that emerge would be substantially different from the (linear) case, where the asynchronicity in the long run is independent from initial conditions (i.e. shocks).

Clearly, when the dynamical system converges towards a point attractor, initial conditions are not essential to explain the long run behavior and

[15]Simon and Velupillai (the latter, one of Goodwin's prized pupils) have worked a great deal on matters concerning the construction of these mechanisms. Simon has focused mainly on the definition of a computational model of human problem solving and Velupillai on the impossibility of constructing mechanisms capable to capture any version of an Olympian model of rationality. To assume that human or artificial agents are able of Olympian performance – i.e. are assumed to be able to compute the uncomputable – is to assume some form of magic, simply because there is no connection whatsoever with actual (past, present or future) humans or with any artificially constructed or to be constructed mechanism. The connection between Simon's *Classical Behavioral Economics* and Velupillai's *Computable Economics* has been discussed and elaborated in several publications. See for example Velupillai (2000, 2010, 2011a); Kao and Velupillai (2012); Velupillai and Zambelli (2011); Zambelli (2010); Kao et al. (2012)

[16]See also the exercise presented in Zambelli (2011b) where all the economies where described with different parameter values and rich dynamics where generated. The model presented here has a similar structure

displacements away from the equilibrium are important only to explain the transition back to the equilibrium [17] .

When the economic systems are oscillators, different initial conditions also determine different long run behavior. In order to make the argument as clear as possible, we have made Fermi-Pasta-Ulam laboratory type experiments where the identical economies have been located on a unidimensional lattice. When perturbed away from their equilibrium values, the economies would converge on different limit cycles. This *path dependency* is the result that was unexpected. Given the transmission of the original shock to the neighboring economies, thorough the changes in trade, we expected the shocks to be dissipated and the system to exhibit, in the long run, total synchronous behavior.

Here we have presented only two examples, the single shock and the sawtooth shocks, but in preparing the paper we have tried many different initial conditions and we have also modified the original model to account for balance of payment adjustments. The qualitative results have in all the different experiments been the same and synchronicity has never been observed.

Richard Goodwin advocated the importance of nonlinearity all his life. The result presented here, most likely, would not have been a surprise to him. Surely, he had understood that dynamical coupling was a complicated matter. Here we have shown that even when the economies are similar, aggregation is not a trivial matter. His intuition that one would need the services of computers was obviously correct.

The importance of initial conditions for the development of the economic system suggests caution in trying to derive general policy rules. A policy can be seen as a particular set of initial conditions. Velupillai has shown the *Impossibility of an effective theory of policy for a complex economy* (Velupillai, 2007, p.284). This does not mean at all that policy is impossible, but quite on the contrary, that policy has to be searched and not determined *a priori* as a one-size-fit-all type prescriptions. What is impossible is to derive *universal* policy rules, but this does not impede the search for a *good* or *satisfiable* policy.

The use of Ulam type laboratory experiments may be very useful in trying to understand the functioning of real economies. These experiments may help train intuition and may allow for the study of new alternatives[18].

[17] This model approach is the one suggested by Frisch (Frisch, 1933) (when erroneously quoting Wicksell, see Velupillai (2008) and Zambelli (2007)), in the famous *Rocking Horse* metaphor: *'If you hit a wooden rocking-horse with a club, the movement of the horse will be very different from that of the club* (Frisch, 1933, p.198)'. This is the model approach used in most of the models like RBC, DSGE and so on. In the same article, Frisch presented also Schumpeter's view - business cycles as being endogenous to the functioning of the capitalistic system of production (see above, footnote 5, p.6)

[18] Some readers may wonder whether computer experiments of the Fermi-Pasta-Ulam

This may enable us to search for determining satisfactory policies.

Goodwin was fully aware of the importance of *' ... the prolonged services of yet unborn calculating machines ...'* (Goodwin, 1947, p.204)[19]. After 60 years, it is very rewarding to be able to use these services that were envisaged and to walk on the path indicated by giants like him.

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type in physics are different from computer experiments in economics because the laws of physics are somewhat more precise than the law of economics. This is not at all the case. The FPU conceptual experiment required the setting of an hypothetical virtual world, where 32 atoms were totally isolated and were located at a precise to the infinitesimal distance - on the string. This is an impossible situation to be reproduced in so-called real systems. A real, i.e. not digital or analog, laboratory experiment could or can only mimic such a condition, but cannot replicate it precisely. The result of their experiment was different from what they had expected. What scholars have learnt from these experiments may be extended, through analogy and induction (see also Keynes (1921), Part.3), to real problems. From the point of view of furthering our understanding, i.e. knowledge, about a problem Ulam type of experiments are not different when applied to economics. Ulam's insight *... such experiments on computing machines would have ... the virtue of having the postulates clearly stated. This is not always the case in an actual physical object or model where the assumptions are not perhaps explicitly recognized.* Replace *physical object economic object* and we have a fruitful methodological suggestion for economists and for algorithmic social scientists.

[19]Goodwin was a good friend of Phillips and was the custodian of the Phillips Monetary National Income Analog Computing Machine at Cambridge. The link between the contribution made by Phillips and that by Goodwin is very strong and is elaborated in several recent papers collected in Velupillai (2011c) (in particular see Kuczynski (2011); Velupillai (2011b); McRobie (2011); Zambelli (2011a) point out that the Phillips machine is actually a flexible accelerator model because the machine does inevitably hit boundaries. Furthermore, Goodwin had been also in close contact with Strotz et al. (1953), who made an analog model of his multiplier-accelerator model.

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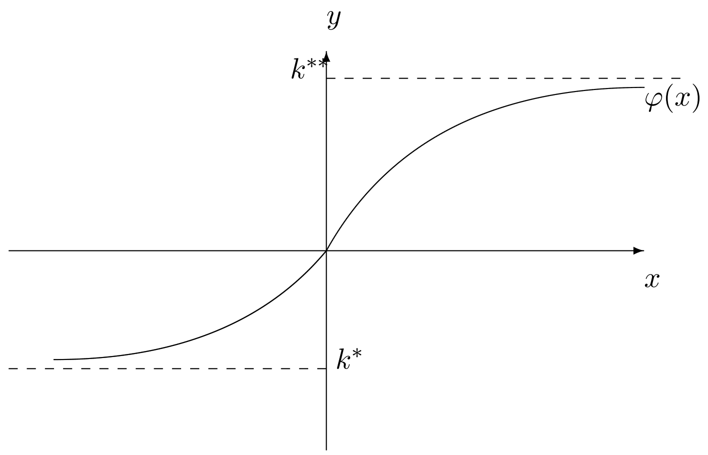
Fig. A.1: *Flexible accelerator (modified)*



A Functional specification of $\varphi(\cdot) = \mathbf{v}_\sigma Y(t) - K(t)$

Equation 2.4 and Fig.2.1 provide a qualitative description of the modified flexible accelerator. In order to conduct our computational experiments we have to give a precise functional form. The flexible accelerator looks roughly like a logistic function, but it is asymmetric because the lower asymptote is not in general in absolute value equal to the higher asymptote. In order to keep the same qualitative behavior we have combined two logistics:

$$y = \varphi(x) = \begin{cases} \frac{2k^{**}}{1+e^{C(x+B)}} & \text{if } x \geq 0 \\ \\ \frac{2k^{*}}{1+e^{C(x+B)}} & \text{if } x < 0 \end{cases} \tag{A.1}$$

where:

$$x = \mathbf{v}_\sigma Y(t) - K(t)$$

$$\omega = \frac{2\mathbf{v}_\sigma (k^{**} - k^{*})}{y(0) - k^{*}}$$

$$C = \frac{\omega^2}{4\mathbf{v}_\sigma (k^{**} - k^{*}) - 2\omega (k^{**} - k^{*})}$$

$$B = \frac{\ln(\frac{k^{**}-k^{*}}{y(0)-k^{*}} - 1)}{C} - x(0)$$

The values for ω, C and B are computed so as to assure that when $x = 0$ and $y = 0$ the derivative of $\varphi(\cdot)$ is maximum and is equal to $\mathbf{v}_\sigma$.

B Parameter values and initial conditions used in the simulations

Economic systems: $i = 1 \dots n$.
Number of economic systems, countries or nations: $n = 32$.
Intercept of the consumption function $\mathbf{C}_{0i} = 1000$
Propensity to consume: $\mathbf{c} = \{c_i = 0.6\}$
Propensity to import: $\mathbf{m} = \{m_i = 0.2\}$
Capital-output ratio: $\mathbf{v}_\sigma = \{v_{\sigma i} = 1.6\}$
Upper value for $\varphi(\cdot)$: $\mathbf{k}_u = \{k_{ui} = 0.2\}$
Lower value for $\varphi(\cdot)$: $\mathbf{k}_l = \{k_{li} = -0.5\}$
Export matrix $\mathbf{\Lambda}$: see Eq. 5.1, p.12.
Step size: $h = \frac{1}{24}$

The differential equations have been numerically approximated following the standard Euler approximation.

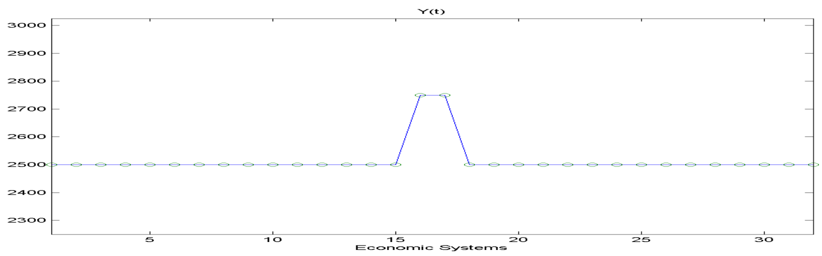


Fig. B.1: **Single Shock** *Initial conditions at t_0 are the equilibrium values for all the variables, but with $Y_{16}(t_0) = 1.1Y_{16}^{eq}$ and $Y_{17}(t_0) = 0.7Y_{17}^{eq}$*

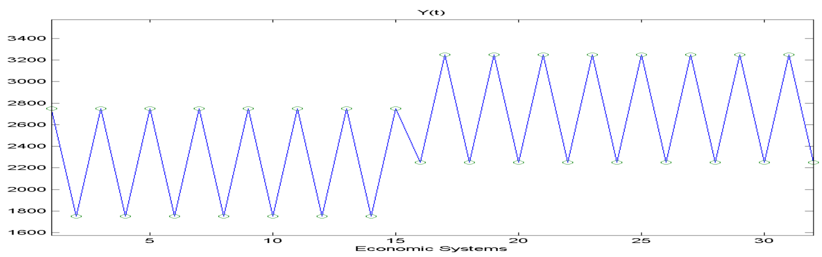


Fig. B.2: **Sawtooth**. *Initial conditions at t_0 are the equilibrium values for all the variables, but with: $Y_i(t_0) = 1.1Y_i^{eq}$ for $i = 1, 3, 5, 7, 9, 11, 13, 15$; $Y_i(t_0) = 0.7Y_i^{eq}$ for $i = 2, 4, 6, 8, 10, 12, 14$; $Y_i(t_0) = 1.3Y_i^{eq}$ for $i = 17, 19, 21, 23, 25, 27, 29, 31$; $Y_i(t_0) = 0.7Y_i^{eq}$ for $i = 16, 18, 20, 22, 24, 26, 28, 30, 32$.*