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CONSTRUCTIVE AND *COMPUTABLE* HAHN-BANACH THEOREMS FOR THE (SECOND) FUNDAMENTAL THEOREM OF WELFARE ECONOMICS[♣]

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[♣] I owe a great debt to my very able and most helpful graduate student, *Ilker Aslantepe* (who should, by all standards of ethical behavior, be a *joint author*), for helping me focus on the constructive and computable aspects of the Hahn-Banach Theorem, when used in the context of the second fundamental theorem of welfare economics. This is a topic that has been at the core of my work in computable and constructive economics for the past three decades. As usual, constant encouragement and ceaseless honest intellectual criticism by my friend and colleague, Professor Stefano Zambelli, has been an inspring force. I am solely responsible for all remaining infelicities.

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Abstract

The Hahn-Banach Theorem plays a crucial role in the second fundamental theorem of welfare economics. To date, all mathematical economics and advanced general equilibrium textbooks concentrate on using non-constructive or incomputable versions of this celebrated theorem. In this paper we argue for the introduction of constructive or computable Hahn-Banach theorems in mathematical economics and advanced general equilibrium theory. The suggested modification would make applied and policy-oriented economics intrinsically computational.

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§1. A Preamble

> "There are *many* – *denumerably*, I suspect – other versions of the [Hahn-Banach] theorem, … . It has many applications not only outside functional analysis but outside mathematics."
> Narici (2007), p. 88; italics added.

The three 'crown jewels' of the mathematical economics of the second half of the twentieth century are undoubtedly the proof of the *existence* of a *Walrasian Exchange Equilibrium* and the mathematically rigorous demonstration of the validity of the *two fundamental theorems of welfare economics*[1]. Brouwer's original fix-point theorem – and a variety of its extensions, primarily that of Kakutani – was instrumental in the classic Arrow-Debreu proof of the former. A variety of variations of the classic *Hahn-Banach theorem* (henceforth referred to as *H-B T*) was used in the formal proof of the latter – again, one could, with considerable doctrine historical legitimacy[2] attribute the use of variants of this mathematical gem to Arrow (1951) and Debreu (1954, 1984, p. 269).

Even within these three 'crown jewels', it is arguably the second fundamental theorem of welfare economics (henceforth referred to as *FTWE II*), that is most fundamental from the point of view of the applied, policy-oriented, economist. For, as Ross Starr (2011, p. 213; italics added) states:

> "The Second Fundamental Theorem of Welfare Economics represents *a significant defense of the market economy's resource allocation mechanism*."

However, every appeal to, or statement of, the H-B T, or its various implications – the most common being in its supporting or separating hyperplane forms – in any of the standard textbooks of general equilibrium theory or mathematical economics[3], are to its *non-constructive* or *incomputable* versions (Arrow-Hahn (1971, p. 382), Stokey-Lucas (1989, pp. 450-1), Bridges (1998, ch. 6 & Appendix C)[4]. In every one of these statements of the H-B T, there is an appeal to one or another non-constructive or incomputable precept. For example, in the Stokey-Lucas

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[1] It is appropriate also that two of our neoclassical founding fathers are associated with the names commonly assigned to these 'crown jewels': *Léon Walras* and *Vilfredo Pareto*.
[2] But see Blaug (2007).
[3] Obviously the caveat '*to the best of this author's knowledge*' is implicitly invoked here.
[4] It will be noted that I have chosen the more standard references, but all of them published *after* Bishop (1967), where a crystal clear constructive version of the H-B T was first given (ibid, pp. 262-3). However, there were, of course, earlier economic classics – for eg., Debreu (1959, ch. 6) and Malinvaud (1953, p. 245) – invoking some variant of the H-B T. There is, surely, no excuse for anyone to invoke a *non-constructive* H-B T after 1967 (and to an *incomputable* H-B T after the appearance of Metakides and Nerode (1982))

allegedly *Geometric form of the Hahn-Banach Theorem*[5] (*ibid*, 450), an appeal is (implicitly) made to Zorn's Lemma and transfinite induction, as does Bridges (p. 263). The caveat, in the Arrow-Hahn use of the H-B T, of its validity in *finite-dimensional spaces* (ibid, p. 382), may *deceive* the unwary reader into thinking that finite counting arguments may obviate the constructive or computable requirements. That this is not so can be discerned from a careful perusal of Schechter (1971, ch. II, §2).

The rest of this paper is structured as follows. In the next section the ε-approximate constructive H-B T is stated and the way it may be used in economics is outlined. A similar task is undertaken for the computable H-B T, as much as possible paralleling Bishop's ε-approximate constructive H-B T.

In the concluding third section a brief outline of Ishihara's version of the exact – i.e., dispensing with the ε-approximation – H-B T is outlined. However, I am not convinced the trade-off for the exact H-B T, in terms of gain in economic intuition, is sufficiently significant from any applicable, policy-oriented point of view for it to be adopted in economic analysis.

Some notes on the ε-approximate, intuitionistic-constructive Brouwer fix-point theorem are also added in the concluding section.

§2. Constructive and Computable Hahn-Banach Theorems

> "Throughout [Computability in Analysis and Physics] we have attempted to give general principles from which the effectivization or noneffectiveness of well-known classical theorems follow as corollaries. .. For example, it would be interesting to have a general principle which gave as corollary the known facts concerning the Hahn-Banach Theorem."
> Pour-El and Richards (1989), p. 194.

[5] Clearly, Stokey-Lucas statement of the H-B T is adapted from Luenberger (1969), §5.11, particularly p. 133, where it is (correctly) attributed to Mazur (1933), without however any specific reference. It was first referred to as the '*Geometric* Hahn-Banach Theorem' by Bourbaki (cf., Narici, *op.cit*, p. 88), whose notion of 'geometry' was singularly non-intuitive, contrary to Luenberger's attempted justification for the name on intuitive grounds. This is not irrelevant in the context of Bishop's constructive version of the H-B T, using also *Brouwerian counterexamples* (see, for example, Mandelkram, 1989), in a crucial way. Luenberger cheerfully acknowledges the use and invoking of Zorn's lemma, which is not even mentioned in Stokey-Lucas (op.cit).

Four constructive and computable versions of the H-B T are now given. The first two, are referred to as Theorems B and I; the next two referred to as M-N 1 and M-N 2, respectively.[6]

> Theorem B (Bishop, 1967, Theorem 4, p. 263; italics added)
> Let λ be a nonzero linear functional on a linear subset V of a *separable* normed linear space B, whose null space – i.e., *kernel* - $N(\lambda)$ is a *located* subset of B. Then for each $\varepsilon>0$ there *exists* a *normable* linear functional ν on B with $\nu(x) = \lambda(x)$ for all x in V, and $\|\nu\| \leq \|\lambda\| + \varepsilon$

Bishop constructs *Brouwerian counterexamples*[7] to demonstrate the necessity of the ε. A particularly illuminating hint for the construction of a relevant Brouwerian counterexample to show the necessity of the ε in Bishop's constructive H-B T is given in Bridges and Richman (1987, p. 46, Problem 19).

Any mathematically trained economist, with a solid grounding in the mathematics of advanced general equilibrium theory would have no difficulty in coming to terms with almost all the concepts used in Theorem B – except *located* and *normable* (perhaps). Furthermore no applied economist, with grounding in mathematical general equilibrium theory would have any reason to accept a *separable* normed linear space. However, it is not at all clear that any standard course in mathematical economics or advanced general equilibrium theory – using one or another of the textbooks cited in the previous section or, indeed, any other – would read '*there exists*' in the above statement of the constructive H-B T in its *constructive sense*.

To be constructively precise, the ordinary operations of *scalar multiplication*, *vector addition* and *norm* have to be *constructively determined*. A *located* subset $N(\lambda)$ of B means it is possible to *constructively calculate* the distance, say $d(x)$, of any point $x \in B$, from $N(\lambda)$; and, a *normable* linear functional (*loc.cit*, p. 249), λ, is defined as one for which $\|\lambda\|$ *exists*, where:

$$\|\lambda| \equiv \text{l.u.b } \{|\lambda(x)| : x \in S\}$$

[6] Referring to the Bishop (B) and Ishihara (I) H-B T results in constructive analysis.; M-N 1 and M-N 2 refer to Metakides and Nerode (1982, 1985) theorems on H-B T, within the framework of computable analysis.

[7] See, again, Mandelkern (1989), especially p. 3 (bold italics, added):

> "Brouwer's critique of .. [classical mathematics showed that] certain classical theorems could not be true, for if they were true, many .. would be trying to use them to ***solve the unsolved*** problems. Such a demonstration is called a *Brouwerian counterexample*; it differs from an ordinary counterexample, which demonstrates that a given statement implies another statement which is known to be false."

Thus, the *constructive* meaning of Theorem B is the following:

> Remark 1
> Given the constructive real number ε, it is possible to constructively calculate ν, and its norm, from a constructive representation of λ, the constructively determined norm of λ, constructively defined vector addition and scalar multiplication in *B*, with its constructive norm, $||.||$.

Ishihara (1989), by restricting the separable normed linear space, showed how Bishop's ε-approximate constructive H-B T can be replaced by one in which the constructive real number ε could be dispensed with. In the restriction, Ishihara uses the notion of a *Gâteaux differentiable* norm and *uniform convexity*[8].

> Theorem I (Ishihara, 1989, p. 80; italics added)
> Let *M* be a linear subset of a *uniformly convex* complete normed linear space *E* with a *Gâteaux differentiable norm* and let *f* be a nonzero normable linear functional on *M*. Then there exists a unique linear functional *g* on *E* such that $g(x) = f(x)$ for all *x* in *M* and $||g||=||f||$.

It goes without saying that *all* of the technical concepts used in the statement of Theorem I are *constructively defined*, in particular the phrase '*there exists*'. However, it does not seem to me to be reasonable to expect an applied or a policy-oriented economist would be persuaded that the gains from introducing new, albeit constructively defined, notions outweigh the intuitive acceptability of the constructive underpinnings of Bishop's ε-approximate H-B T.

Turning next to computable Hahn-Banach Theorems, there are two, almost exactly parallel results, to the above two by Bishop and Ishihara for *constructive analysis*, developed by Metakides, Nerode (*op.cit*), for *computable analysis*.

> Theorem M-N 1(Metakides & Nerode, 1982, p. 329 & Metakides & Nerode, 1985, p. 87)
> Let λ be a non-zero *computably continuous* linear functional on a linear subset of a *computable* Banach space *B* whose null space $N(\lambda)$ is a *computably located* subset of *B*. Then for any $\varepsilon > 0$, there is a *computably* continuous linear functional ν on *B* with *computable* norm such that $\nu(x) = \lambda(x)$ for all $x \in V$, and $||\nu|| \leq ||\lambda|| + \varepsilon N(\lambda)$ is *computably* located precisely when λ has a *recursive* norm[9].

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[8] See D.L. Johns and C.G. Gibson (1981), where these concepts were first used in the constructive analysis of duality in Orlicz Spaces.
[9] After reading Soare (2013), I have tended to replace every use of 'recursive' with 'computable'!

It is clear that one replaces *constructive* in Theorem B, with *recursive* in Theorem M-N 1, to get almost the same result – except that in this case one has a *computable*, rather than a *constructive* H-B T. Even Brouwerian counterexamples, of the kind used in Bishop, are applicable in the computable version of the H-B T. The only other care that has to be taken in interpreting the equivalences between the two theorems is that in Brouwerian constructivism all functions are uniformly continuous, which is not the case in computability theory.

The computable equivalent of Theorem I is the following.

> Theorem M-N 2 (Metakides & Nerode, 1985, p. 87)
> The computable Hahn-Banach theorem holds for *finite* dimensional computably presented Banach spaces *without* the ε.

Two remarks are in order, here.

> Remark 2
> Because of the finite dimensionality of the above theorem, Bishop-type Brouwerian Counterexamples are not valid; a computably presented Bishop-type, Brouwerian counterexample, in the computable H-B T, is valid only if the Banach space is infinite dimensional.

> Remark 3
> The Arrow-Hahn restriction of the validity of the *classical* H-B T to finite dimensionality (Arrow and Hahn, op.cit., p. 382) is *non-constructive* and *incomputable* (see above). M-N-2 is, however, exactly tailored for the applied and policy-oriented economist, who might wish to *compute*[10] efficient allocations, in relation to a particular exchange equilibrium.

§3. Concluding Notes

> " … [L]et us return to the issue of computation over the reals as a foundation for *scientific computation*, aka *computational science* or *numerical analysis*. …
> [I]n practice, those computations are made[11] in 'floating point arithmetic' using finite decimals with relatively few significant digits, for which computation *per se* simply reduces to computation over rational numbers."
> Feferman (2013, p. 56-7); italics in the original.

This being so, why do we theorize – in mathematical economics and general equilibrium theory, in particular – using one kind of mathematics and unconsciously rely on another kind for the justification of rigorous computation? Why are students not taught, *ab initio*, constructive and

[10] The *proof* of M-N 2 is, however, *non-constructive*.
[11] Obviously utilizing a digital computer; only trivial modifications to Feferman's profound observation will have to be made for it to be valid also for analogue or hybrid computation.

computable analysis, instead of conventional real analysis, replete as it is, with undecidabilities, uncomputabilities and non-constructivities?

This author has no simple-minded answer to this almost trivial question – except that he, himself, does not teach economic theory in the mathematical mode within the framework of classical real analysis, founded on axiomatic set theory. In particular he does not appeal to any form of the axiom of choice (or Zorn's Lemma).

This author learned his set theory from the monumental book by Kuratowski and Mostowski (1976). The caveat in the penultimate paragraph of the Preface to the First Edition of that remarkable book has remained etched in his increasingly frail memory:

> "In order to illustrate the role of the axiom of choice we marked by a small circle ^{o} all theorems in which this axiom is used."

Would that economists were as scrupulous as these two famous authors!

Had we, as students of mathematical economists or general equilibrium theory, been cautioned that any appeal to the H-B T was in its non-constructive or incomputable formalism, perhaps we may have wondered, timid as we were as graduate students, in awe of our obviously more learned teachers, even aloud, whether there were not constructive or computable alternatives of these celebrated theorems – especially for applications to implement the second fundamental theorem of welfare economics in a policy context.

This, however, would be predicated upon us, as graduate students, being at least having a nodding acquaintance with constructive or computable analysis, in addition to classical real analysis.

At the very beginning of this paper it was pointed out that the three crown jewels of the mathematical economics of the second half of the twentieth century were provided with rigorous mathematical demonstrations, for their formal validity, by the use of two of the most celebrated theorems of analysis: the Brouwer fix-point theorem and the H-B T. As we conclude, it is clear that it is possible to use constructive or computable versions of the latter, provided the economic formalism is appropriately adapted to the new frameworks of constructive and computable analyses.

Even if this 'research program' was successfully implemented, it may remain slightly 'stunted' without a similar reformulation of the Brouwer fix-point theorem. Is this possible?

Happily, there is a partial, affirmative, answer. Brouwer did reformulate his original *non-constructive* fix-point theorem and provided an intuitionistc-constructive version of it (Brouwer, 1952). An *ε-approximate*, *constructive*, fix point theorem – uncannily similar to Theorem B - was developed, for which the celebrated Sperner Lemma can be applied for the practical construction of an algorithm – thus uniting it with computability theory (van Dalen, 2011).

However, before this remarkable intuitionistic-constructive fix point theorem could be applied to compute a Walrasian (or Arrow-Debreu) equilibrium – within the research program of computable general equilibrium theory– the *Uzawa equivalence theorem* (Uzawa, 1962) will need to be constructivized or effectivized. This is a provably impossible task, because *excess demand functions are impossible to effectivize* (cf. Velupillai, 2013). This *impossibility* was an answer to the characteristically prescient question posed by the doyen of twentieth century mathematical economics and general equilibrium theory, Kenneth Arrow (1986; italics added):

> "[T]he claim the *excess demands are not computable* is a much profounder question for economics that the claim that equilibria are not computable. The former challenges economic theory itself; if we assume that human beings have calculating capacities not exceeding those of Turing machines, then the *non-computability of optimal demands is a serious challenge to the theory* that individuals choose demands optimally."

In this author's opinion, this impossibility result is of greater significance for mathematical economics and the many claims of general equilibrium theory – and, these days, even of careless claims on the feasibility of microfounded macroeconomics - than the much vaunted Sonnenschein-Debreu-Mantel theorem on the aggregation of excess demand functions. The similarity, therefore, between the two ostensibly similar ε-approximate theorems – between Theorem B and Brouwer's intuitionistically-constructive fix point theorem (van Dalen, *op.cit*) – remain illusory. This illusion can, of course, be dispelled within the framework of *Computable Economics*.

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