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ALGORITHMIC ECONOMICS: INCOMPUTABILITY, UNDECIDABILITY AND UNSOLVABILITY IN ECONOMICS

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Algorithmic Economics: Incomputability, Undecidability and Unsolvability in Economics[1]

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Dedicated to **Martin Davis** for introducing me to '*the pleasures of the illicit*'[2].

> "Being unfamiliar with what counts as a problem in economics I am astounded that *decidability and other foundational problems* should turn up here. Quantum mechanics is pretty OK; there are problems, true, but there are also lots of very precise predictions. I have not noticed anyone in the field went into the foundations of mathematics to get things going. *Why do these matters turn up in economics?*"
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> **Paul Feyerabend,** *Letter to the Author*, '02/04/92' (4th February, 1992); italics added.

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[1] I am greatly indebted to the late Barry Cooper, dear friend and a fellow Turing *aficionado* for inviting me, an outsider and an outlier *par excellence*, to contribute to the *Proceedings of the Turing Centennial* conference, held at *Chicheley Hall,* in June, 2012. The occasion of the conference itself was both a pleasure and an honour to attend as an invited speaker. It also gave me a chance, finally, to meet and get to know, personally, some of the giants of 20th century computability theory, in particular, *Martin Davis*, *Yuri Matiyasevich* and *Robert Soare*. I have benefited greatly from the friendly suggestions for improvement by an anonymous referee and the incredibly good-humoured patience shown by Mariya Soskova. In the time since June, 2012, when a first draft of this paper was composed, my own knowledge on computability theory, constructive mathematics and Alan Turing's writings have been deepened; I hope that is also reflected in this revision.

[2] *Preface* to Davis (6).

Abstract

Economic theory, like any theoretical subject where formalised deduction is routinely practised, is subject to recursive, or algorithmic, incomputabilities, undecidabilities and unsolvabilities. Some examples from core areas of economic theory, micro, macro and game theory are discussed in this paper. Some background to the nature of the problems that arise in economic decision theoretic contexts is given, before presenting some results, albeit largely in a sketchy form.

§1. By way of A Prologue on *Theses*[3]

> "The last of the original three [i.e, *λ-definability*, *General Recursiveness* and Turing Machines] equivalent *exact* definitions of *effective calculability* is *computability by a Turing Machine*. I assume my readers are familiar with the concept of a Turing Machine … .
> In a conversation at San Juan on October 31, 1979, [Martin] Davis expressed to me the opinion that the equivalence between Gödel's definition of general recursiveness and mine … and my normal form theorem, were considerations that combined with Turing's arguments to convince Gödel of the *Church-Turing thesis*."
> Kleene (17), pp. 61-2; italics added.

'Why do these matters', of (in)computability, (un)decidability and (un)solvability, 'turn up', in economics, as Paul Feyerabend posed it, in his letter to me[4]. I think, however, Feyerabend is not quite correct on 'not ... anyone in the field [of quantum mechanics going] into the foundations of mathematics to get things going.' Karl Svozil (32), with copious references to the pre-1992 results, David Finkelstein (11) and Roger Penrose (20, 21) are obvious counter-examples to this view. I would like to add, further, notwithstanding the general vision of this paper, 'there are lots of very precise predictions', in economics, even if not founded on orthodox economic theory.

Today, two decades, and more, later, if it is possible to give a guarded, even unequivocal answer, it is because the concepts and foundations of economic theory are increasingly given a computational basis, founded on computability theory, constructive mathematics or even intuitionistic logic-based smooth infinitesimal analysis (thus dispensing with any use of the *tertium non datur*, in proofs involving uncountable infinities). Even classic economic problems invoke *computer aided proofs* - for example in the travelling salesperson's problem or in the use of the Poincaré-Bendixson theorem for proving the existence of fluctuations in macro dynamical models, algorithmic procedures are routinely invoked. Even in a domain of classical game theory, Max Ewe's (9) demonstration of the 'existence' of a min-max solution to a two-person game and Michael Rabin's pioneering work on arithmetical games (Rabin, 23), foundational issues of a mathematical nature arise in an economic setting.

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[3] The paradigmatic example of which is, of course, the Church-Turing *Thesis*, but there are two others I should mention in this prologue: *Brattka's Thesis* (Brattka, 4) and what I have called the *Kolmogorov-Chaitin-Solomonoff Thesis* (Velupillai, 41). I should have added the name of Per Martin-Löf to the latter trio.

[4] Cited above, on the title page.

In other writings, for example Velupillai (43, p. 34), I am on record as saying:

> "The three 'crown jewels' of the mathematical economics of the second half of the twentieth century are undoubtedly the proof of the *existence of of a Walrasian Exchange Equilibrium* and the mathematically rigorous demonstration of the validity of the *two fundamental theorems of welfare economic*s."

Unfortunately, in orthodox mathematical economics these important theorems are proved non-computationally[5]. For example, in Brainard & Scarf (3, p. 58), we read:

> "But we know of no argument for the existence of equilibrium prices in this restricted model that does not require the full use of *Brouwer's fixed point theorem*. Of course fixed point theorems were not available to Fisher .. ."

This claim, was made as late as only about ten years ago, despite Smale's important point, made thirty years before Brained & Scarf (op.cit) in Smale, 29, p. 290 (italics added):

> The existence theory of the static approach is deeply rooted to the use of mathematics of fixed point theory. This one step in the liberation from the static point of view would be to *use a mathematics of a different kind*.
>
> ...
>
> I think it is fair to say that for the main existence problem in the theory of economic equilibria, one can now bypass the fixed point approach and attack the equations directly to give t*he existence of solutions, with a simpler kind of mathematics and even mathematics with dynamic and algorithmic overtones*."

As for the two fundamental theorems of welfare economics, non-constructive, non-algorithm, versions of the Hahn-Banach Theorem(s) are invoked in the proofs (particularly of the more important second fundamental theorem of welfare economics).

I need to add here[6] that very little work in economics in the mathematical mode is done with models of computable reals, nor are - to the best of my knowledge - algorithms in economics

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[5] By 'computational' I mean either computably or constructively - i.e., algorithmically; hence, 'non-computationally' is supposed to mean 'non-algorithmically'. The above statement applies also to all standard results of orthodox game theory, despite occasional assertions to the contrary. Ewe, as mentioned above, is the exception.

[6] I am greatly indebted to an anonymous referee for raising the relevance of this point here. I have dealt with the issue in many of my writings on algorithmic economics of the last decade or so. The trouble in economics is that using reals or approximations is indulged in blind and *ad hoc* ways. Even the approximations involved in either discretising continuous dynamical systems - however elementary, but nonlinear - or computing approximate Nash or Arrow-Debreu-Walrasian equilibria, are done carelessly.

specified in terms of *interval analysis*. Of course, neither *smooth infinitesimal analysis* nor constructive formalisms are routine in economics (Bridges, 5, is a noble exception), claims to the contrary (example, Mantel, 18).

In this paper there is an attempt to use a 'simpler kind of mathematics and even with dynamic and algorithmic overtones', even if the results appear to be 'negative' solutions; it must be remembered that there are obvious positive aspects to negative solutions - one doesn't attempt to analyse the impossible, or construct the infeasible, and so on (hence the importance of theses, which is, after all, what the second 'law'[7] of thermodynamics is).

Computable General Equilibrium[8] theory, *computational* economics, agent based *computational* models, *algorithmic* game theory are some of the frontier topics in core areas of economic theory and applied economics. There is even a journal with the title *Computational Economics*[9].

However, very few in economics seem to pay attention to the notion of a *thesis* – such as the Church-Turing *Thesis* – and, therefore do not pay sufficient care to the importance of tying a concept developed by *human intuition* to a *formal* notion that is claimed to encapsulate that intuitive notion *exactly*[10]. That the Church-Turing *Thesis* is a result of this kind of identification, between the intuitive notion of *effective calculability* and the formal notions of – independently developed – *general recursiveness*, *λ-definability* and *Turing Machines* is not fully appreciated by the mathematical economics community.
As a result of this particular kind of disinterest, economic theory in its mathematical mode forces economic concepts to conform to independently developed mathematical notions, such

[7] It must be remembered that Emil Post refereed to 'theses' as 'natural laws'.
[8] The foundations on which the much vaunted recursive competitive equilibrium (RCE) is based, from which, via, real business cycle (RBC) theory, the frontier, fashionable, models of dynamic stochastic general equilibrium (DSGE) framework is carved out. None of these models are computable or recursive in any of the formal senses of computability theory, nor in any version of constructive mathematics. In Velupillai (44) I discuss the non-algorithmic aspects of these problems in greater detail, from either a computable or a constructive point of view.
[9] One of whose associate editors once wrote me – when inviting me to organize a session in the annual event sponsored by the Journal – that 'we compute the uncomputable'. He was, with some seriousness, referring to the fact that the overwhelming majority of computational economists are blissfully ignorant of the computability theory underpinnings of whatever it is they compute. Nor are they seriously interested in the link between dynamical systems, numerical analysis – sometimes referred to as 'scientific computing'- and computability theory (cf. Stuart & Humphries, 31).
[10] I use this word in the precise sense in which it is invoked by Kleene, in the above quote.

as continuity, compactness, and so on. It is – to the best of my knowledge – never acknowledged that there are intuitive notions of *continuity* which cannot be encapsulated by, for example, the concept of a *topological space* (Gandy, 13, p. 73).

A *thesis* is not a *theorem*. The Church-Turing Thesis came about, as suggested above, as a result of trying to find a *formal* encapsulation of the *intuitive* notion of *effective calculability*. What is the difference between a *Thesis* and a *Theorem*? Perhaps one illuminating way to try to answer this question is to reflect on `an imaginary interview between a modern mathematician [Professor X] and ... Descartes', devised by Rosser (24, pp. 2-3), to illustrate the importance of the open-ended nature of any claimed *exact* equivalence between an intuitive concept and a formal notion:

> "… Descartes raised one difficulty which Professor X[11] had not foreseen. Descartes put it as follows:
> 'I have here an important concept which I call continuity. At present my notion of it is rather vague, not sufficiently vague that I cannot decide which curves are continuous, but too vague to permit of careful proofs. You are proposing a precise definition[12] of this same notion. However, since my definition is too vague to be the basis for a careful proof, how are we going to verify that my vague definition and your precise definition are definitions of the same thing?
> If by 'verify' Descartes meant 'prove,' it obviously could not be done, since his definition was too vague for proof. If by 'verify' Descartes meant 'decide,' then it might be done, since his definition was not too vague for purposes of coming to decisions. Actually, Descartes and Professor X did finally decide that the two definitions were equivalent, and arrived at the decision as follows. Descartes had drawn a large number of curves and classified them into continuous and discontinuous, using his vague definition of continuity. He and Professor X checked through all these curves and classified them into continuous and discontinuous using the ε - δ definition of continuity. Both definitions gave the same classification. As these were all the interesting curves that either of them had been able to think of, the evidence seemed 'conclusive' that the two definitions were equivalent."

How did they come to this conclusion? By comparing the classifications into continuous and discontinuous all those `interesting curves' either of them could 'think of', using their own respective definitions - the intuitive and the (so-called) precise - and finding they resulted in identical characterisations. Thus, 'the evidence seemed "conclusive" that the two definitions were equivalent' (*ibid,* p.3). The *evidence for equivalence can only 'seem' conclusive*.

[11] Rosser's explanation for this particular 'christening' of the 'modern mathematician was (*ibid*, p. 1): "[I]n the classic tradition of mathematics we shall refer to him as Professor X."
[12] The 'proposal' by Professor X was the familiar 'ε-δ' definition of continuity.

Any and every computation that is implementable by a Turing Machine answers *all* such questions of the 'equivalence' between 'intuitive' notions of effective calculability' and formal definitions of computability unambiguously: every model of computation thus far formally defined (going beyond the triple noted by Kleene, above) - Turing Machines, Post's Machine, Church's λ-Calculus, General Recursiveness, the Shepherdson-Sturgis Register Machines, etc., - is formally equivalent to any other[13]. But this kind of categorical assertion requires me to assume a framework in which the Church-Turing Thesis is assumed. This is not so in, for example, Brouwerian constructive mathematics, where, nevertheless, *all* functions are continuous; *ditto* for *smooth infinitesimal analysis*, which is founded upon a kind of *intuitionistic logic*.

As summarised by the classic and original definition of this concept by Kleene (16, pp. 300-1):

- Any general recursive function (predicate) is effectively calculable.
- Every effectively calculable function (effectively decidable predicate) is general recursive.
- The Church-Turing Thesis is also implicit in the conception of a computing machine formulated by Turing and Post.

And, Kleene went on (*ibid*, pp. 317-8; italics added):

> "Since our original notion of effective calculability of a function (or of effective decidability of a predicate) is a somewhat *vague intuitive* one, the thesis cannot be proved.
> The *intuitive notion* however is real, in that it vouchsafes as effectively calculable many particular functions, .. and on the other hand *enables us to recognize that our knowledge about many other functions is insufficient to place them in the category of effectively calculable functions*."[14]

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[13]It is, of course, this that is stressed by Gödel when he finally accepted the content of the Church-Turing Thesis (Gödel,14, p.84; italics added):

> "It seems to me that [the] importance [of Turing's computability] is largely due to the fact that with this concept one has for the first time succeeded in giving an *absolute definition* of an interesting epistemological notion, i.e., one *not depending on the formalism chosen*."

[14] It is, surely, the method adopted by Ramanujan, a further step down the line of mathematical reasoning:

> "[I]f a significant piece of *reasoning* occurred somewhere, and the total mixture of evidence and intuition gave [Ramanujan] *certainty*, he looked no further."
>
> Hardy, 15, p. 147; italics in the original.

Once economic theoretical concepts are underpinned by computability or constructive theoretical formalisations, and once computation is itself considered on a reasonably equivalent footing with traditional analytical methods, then, it is *inevitable* that decidability[15], computability and solvability issues - almost all in a recursive sense - will rise to the forefront.

The computable approach to the mathematisation of economics, *enables us to recognize that our knowledge about relevant functions is insufficient to place them in the category of effectively calculable functions*. This 'insufficiency' and its formal 'recognition' is what enables one to derive undecidable, incomputable and unsolvable problems in economics - but also to find ways to decide the undecidable, compute the incomputable and solve the unsolvable..

§2. A Menu of Undecidable, Uncomputable and Unsolvable Problems in Economics

> "Indeed virtually any 'interesting' question about dynamical systems is – in general – *undecidable*.
> This does not imply that it cannot be answered for specific system of interest:….[I]t does demonstrate the *absence of any general formal technique*."
> Stewart, 30, p. 664; italics added.

§2.1 The Undecidability of the Excess Demand Function

On 21st July, 1986, Arrow wrote as follows to Alain Lewis (Arrow Papers, Box 23; italics added):

> "[T]he claim the *excess demands are not computable* is a much profounder question for economics than the claim that equilibria are not computable. The former challenges economic theory itself; if we assume that human beings have calculating capacities not exceeding those of Turing machines, then the *non-computability of optimal demands is a serious challenge to the theory* that individuals choose demands optimally."

That 'the excess demands are not computable' – in the dual, *undecidable*, form suggested by Rózsa Péter (*op.cit*) - can be shown using *one* of Turing's enduring results – the *Unsolvability*

[15] The 'duality' between effective calculability – i.e., computability - and effective undecidability, made explicit by the Church-Turing Thesis is described with characteristic and concise elegance by Rózsa Péter (Péter, 22, p. 254; italics in the original):

> "One of the most important applications of the [Church-Turing] thesis, making precise the concept of effectivity is the proof of the *effective undecidability of certain problems*."

of the Halting Problem for Turing Machines[16]. On the other hand, it can also be proved without appealing to this result (although the phrase 'not exceeding those of Turing machines' is not unambiguous).

To prove that the excess demand functions are effectively undecidable, it seems easiest to start from one half of the celebrated *Uzawa Equivalence Theorem* (Uzawa, 40). This half of the theorem shows that the *Walrasian Equilibrium Existence Theorem* (**WEET**) implies the *Brouwer fix point theorem* and the finesse in the proof is to show the feasibility of devising a continuous excess demand function, ***X***(***p***), satisfying *Walras' Law* (and *homogeneity*), from an arbitrary continuous function, say ***f***(.): ***S***→***S***, such that the equilibrium price vector implied by ***X***(***p***) is also the fix point for ***f***(.), from which it is 'constructed'. The key step in proceeding from a given, arbitrary, ***f***(.): ***S***→***S*** to an excess demand function ***X***(***p***) is the definition of an appropriate scalar:

$$\mu(p) = \frac{\sum_{i=1}^{n} p_i f_i[\frac{p}{\lambda(p)}]}{\sum_{i=1}^{n} p_i^2} = \frac{p.f(p)}{|p|^2} \tag{1}$$

Where:

$$\lambda(p) = \sum_{i=1}^{n} p_i \tag{2}$$

From (1) and (2), the following excess demand function, $X(p)$, is defined:

$$x_i(p) = f_i(\frac{p}{\lambda(p)}) - p_i\mu(p) \tag{3}$$

i.e.,

$$X(p) = f(p) - \mu(p)p \tag{4}$$

[16] The anonymous referee has pointed out, most perceptively, that 'there are many 'specific' Turing machines for which the associated Halting problem 'is' [algorithmically] decidable.' Absolutely true - a result known for at least the past four or five decades. It is fairly 'easy' to establish *criteria for the solvability of the Halting problem* thereby, also, showing that the relevant Turing Machine is *not* universal. For simplicity, in the above discussion I shall work only with Turing Machines that do not satisfy any of the known criteria for the 'solvability of the Halting problem.' I have to add two caveats to this in the context of the problem of the computability of the *excess demand function*. First of all, the claims in orthodox theory, for the validity of the excess demand function, are 'universal'; essentially, that one is working with Universal Turing Machines. Secondly, the kind of functions that should be used as *excess demand function* should be constructive or, at least, functions of the *smooth infinitesimal* type.

It is simple to show that (3) [or (4)] satisfy:

(i). $\boldsymbol{X}(\boldsymbol{p})$ is continuous for all prices, $\boldsymbol{p} \in \boldsymbol{S}$

(ii). $\boldsymbol{X}(\boldsymbol{p})$ is homogeneous of degree 0;

(iii). $\boldsymbol{p} \bullet \boldsymbol{X}(\boldsymbol{p}) = 0, \forall \boldsymbol{p} \in$ S, i.e., Walras' Law holds:

$$\sum \boldsymbol{p}_i \boldsymbol{x}_i(\boldsymbol{p}) = \boldsymbol{0}, \qquad \forall p \in S \ \& \ \forall i = 1.....n \qquad (5)$$

Hence, $\exists \boldsymbol{p}^*$ s.t., $\boldsymbol{X}(\boldsymbol{p}^*) \leq 0$ (with equality unless $\boldsymbol{p}^*=0$). Elementary logic and economics then imply that $\boldsymbol{f}(\boldsymbol{p}^*)=\boldsymbol{p}^*$. I claim that the procedure that leads to the definition of (3) [or, equivalently, (4)] to determine $\boldsymbol{p}^*$ is provably *undecidable*. In other words, the crucial scalar in (1) cannot be defined recursion theoretically to *effectivize* a sequence of projections that would ensure convergence to the equilibrium price vector.

Theorem 1:

$\boldsymbol{X}(\boldsymbol{p}^*)$, as defined in (3) [or (4)] above is *undecidable*; i.e., cannot be determined algorithmically.

Proof:

Suppose, contrariwise, there is an algorithm which, given an arbitrary $\boldsymbol{f}(.)$: $\boldsymbol{S} \rightarrow \boldsymbol{S}$, determines $\boldsymbol{X}(\boldsymbol{p}^*)$. This means, therefore, in view of (i)-(iii) above, that the given algorithm determines the equilibrium $\boldsymbol{p}^*$ implied by WEET. In other words, given the arbitrary initial conditions $\boldsymbol{p} \in \boldsymbol{S}$ and $\boldsymbol{f}(.)$: $\boldsymbol{S} \rightarrow \boldsymbol{S}$, the assumption of the existence of an algorithm to determine $\boldsymbol{X}(\boldsymbol{p}^*)$ implies that its *halting configurations* are *decidable*. But this violates the ***undecidability of the Halting Problem for Turing Machines***. Hence, the assumption that there exists an algorithm to determine - i.e., to construct – $\boldsymbol{X}(\boldsymbol{p}^*)$ is untenable.

Remark 1:

Alas, the proof is uncompromisingly non-constructive – i.e., the proposition is, itself, established by means of appeals to noneffective methods. This is, I believe, an accurate reflection of a perceptive observation made by Davis, Matiyasevic and Robinson (8 p. 340):

> "[T]hese [*reductio ad absurdum* and other noneffective methods] may well be the best available to us because *universal methods for giving all solutions* do not exist."

This is a strengthening of the point made by Stewart (*op.cit*), on the absence of 'general formal techniques', to find *all* solutions to any given problem (or any solution to *all* given problems of a given type), leads to undecidabilities, incompatibilities or unsovabilities.

Modesty and humility in the posing of solvable problems, decidable issues and computable entities is a virtue.

§2.2 Uncomputability of Rational Expectations Equilibria

There are two crucial aspects to the notion of rational expectations equilibrium - henceforth, *REE*: an individual optimization problem, subject to perceived constraints, and a system wide, autonomous, set of constraints imposing consistency across the collection of the perceived constraints of the individuals. The latter would be, in a most general sense, the accounting constraint, generated autonomously, by the logic of the macroeconomic system. In a representative agent framework the determination of *REE*s entails the solution of a general fix point problem. Suppose the representative agent's perceived law of motion of the macroeconomic system (as a function of state variables and exogenous `disturbances') as a whole is given by $\boldsymbol{H}$[17]

The system wide autonomous set of constraints, implied, partially at least, by the optimal decisions based on perceived constraints by the agents, on the other hand, imply an actual law of motion given by, say, $\boldsymbol{H}^0$. The search for fixed-points of a mapping, $\boldsymbol{T}$, linking the individually perceived macroeconomic law of motion, $\boldsymbol{H}$, and the actual law of motion, $\boldsymbol{H}^0$ is assumed to be given by a general functional relationship subject to the standard mathematical assumptions:

$$\boldsymbol{H}^0 = \boldsymbol{T}(\boldsymbol{H}) \qquad (6)$$

Thus, the fixed-points of $\boldsymbol{H}^*$ of $\boldsymbol{T}$, in a space of functions, determine *REE*s:

$$\boldsymbol{H}^* = \boldsymbol{T}(\boldsymbol{H}^*) \qquad (7)$$

Suppose $\boldsymbol{T}: \boldsymbol{H} \rightarrow \boldsymbol{H}$ is a recursive operator (or a recursive program $\boldsymbol{\Gamma}$). Then there is a computable function $\boldsymbol{f}_t$ that is a *least fixed point* of $\boldsymbol{T}$:

$$\mathrm{T}(\boldsymbol{f}_t) = \boldsymbol{f}_t \qquad (8)$$

$$\text{If } \boldsymbol{T}(\boldsymbol{g}) = \boldsymbol{g}, \text{ then } \boldsymbol{f}_t \sqsubseteq \boldsymbol{g} \qquad (9)$$

This result can be used directly to show that there is a (recursive) program that, under any input, outputs exactly itself. It is this program that acts as the relevant reaction or response

[17] Readers familiar with the literature will recognise that the notation $\boldsymbol{H}$ reflects the fact that, in the underlying optimisation problem, a ***Hamiltonian*** function has to be formed.

function, ***T***, for an economy in REE. However, finding this particular recursive program, by specifying a dynamical system leads to the noneffectivity of REE, in a dynamic context[18] Hence, the need for learning processes to find this program, unless the theorem is utilized in its constructive version. Thus, the uncomputability of ***REE***:

Theorem 2:

No dynamical system can effectively generate the recursive program ***Γ***.

Proof:

A trivial application of ***Rice's Theorem*** - in the sense that any nontrivial property of a dynamical system - viewed algorithmically - is undecidable. The intended interpretation is that only the properties of the empty set and the universal set can be effectively generated.

Remark 2:

Given the way Rice's Theorem is proved, the same remarks as above, in Remark 1, apply.

§2.3 Algorithmically Unsolvable Problems in Economics

> "No mathematical method can be *useful* for *any* problem *if it involves much calculation*."
> Turing (38), p. 9; italics added.

Although it is easy to illustrate unsolvability of economic problems in the same vein as adopted in the previous two subsections, I shall not take that approach. Instead, this subsection will be an introduction to Herbert Simon's vision on *Human Problem Solving* Newell & Simon, 19), in the light of Turing's approach to *Solvable and Unsolvable Problems* (Turing, 38).

Turing's fundamental work on *Solvable and Unsolvable Problems* (Turing, 38), *Intelligent Machinery* (Turing, 35) and *Computing Machinery and Intelligence* (Turing, 36) had a profound effect on the work of Herbert Simon, the only man to win both the *ACM Turing Prize* and the *Nobel Memorial Prize in Economics*, particularly in defining *boundedly rational* economic agents as *information processing systems* (IPS) solving decision problems[19].

A comparison of Turing's classic formulation of *Solvable and Unsolvable Problems* and Simon's variation on that theme, as *Human Problem Solving* (Newell & Simon, 1972), would

[18] Proving the noncomputability of a *static* ***REE*** is as trivial as proving the uncomputability of a *Walrasian* or a *Nash* equilibrium.

[19] In the precise sense in which this is given content in mathematical logic, metamathematics, computability theory and model theory.

be an interesting exercise, but it must be left for a different occasion. This is partly because the *human problem solver* in the world of Simon needs to be defined in the same way Turing's approach to *Solvable and Unsolvable Problems* was built on the foundations he had established in his classic of 1936-37.

It is little realised that four of what I call the *Six Turing Classics – On Computable Numbers* (Turing 33), *Systems of Logic Based on Ordinals* (Turing, 34), *Proposal for Development in the Mathematics Division of an Automatic Computing Engine* (ACE) in 1946, published in Turing (39), *Computing Machinery and Intelligence* (Turing, 36), *The Chemical Basis of Morphogenesis* (37) and *Solvable and Unsolvable Problems* (Turing, 38) – should be read together to glean *Turing's Philosophy*[20] *of Mind*. Simon, as one of the acknowledged founding fathers of computational cognitive science was deeply indebted to Turing in the way he tried to fashion what I have called *Computable Economics* (Velupillai, 41). It was not for nothing that Simon warmly acknowledged – and admonished – in his essay in a volume 'memorializing Turing' (Simon, 27, p. 81), titled *Machine as Mind*[21]:

> "If we hurry, we can catch up to Turing on the path he pointed out to us so many years ago."

Simon was on that path, for almost the whole of his research life.

Building a Brain, in the context of economic decision making, meant a mechanism for encapsulating human intelligence, underpinned by rational behaviour in economic contexts. This was successfully achieved by Herbert Simon's lifelong research program on computational behavioural economics[22].

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[20] Remembering Feferman's (10, p. 79) cautionary note that Turing never tried to develop an over- all *philosophy of mathematics* ...', but not forgetting that his above works were decisive in the resurrection of a particular vein of research in the philosophy of mind, particularly in its cognitive, neuroscientific, versions pioneered by Simon.

[21] To which he added the caveat (*ibid*, p. 81):

> "I speak of 'mind' and not 'brain'. By mind I mean a system [a mechanism] that produces thought .. .'

I have always interpreted this notion of 'mechanism' with Gandy's *Principles for Mechanisms* (Gandy, 12) in *mind* (sic!).

[22] I refer to this variant of behavioural economics, which is underpinned by a basis in *computational complexity theory*, as *classical* behavioural economics, to distinguish it from currently orthodox behavioural economics, sometimes referred to as *modern* behavioural economics, which has no computational basis whatsoever.

From the early 1950s Simon had empirically investigated evidence on human problem solving and had organised that evidence within an explicit framework of a theory of sequential information processing by a Turing Machine. This resulted in (Simon, 26 p. x; italics added):

> "[A] general theory of human cognition, not limited to problem solving, [and] a methodology for expressing *theories of cognition as programs* [for digital computers] and for using [digital] computers [in general, Turing Machines] to simulate *human thinking*.

This was the first step in replacing the traditional *Rational Economic Man* with the computationally underpinned *Thinking* i.e., Intelligent - Man. The next step was to stress two empirical facts (ibid, p. x; italics added):

i. 'There exists *a basic repertory of mechanisms and processes* that Thinking Man uses in all the domains in which he exhibits *intelligent behaviour*."
ii. "The models we build initially for the several domains must all be assembled from this same basic repertory, and common principles of architecture must be followed throughout."

It is easy to substantiate the claim that the *basic repertory of mechanisms and processes* are those that define, in the limit, a *Turing Machine* formalisation of the Intelligent Man, when placed in the decision-making, problem-solving, context of economics (cf. Velupillai, 2010).

However, the unsolvability of a problem, implied in any Turing Machine formalization of decision processes, does not really stop people from looking for a solution for it, particularly not Herbert Simon. Sipser (1997) admirably summarises the *pros* and *cons* of proving the unsolvability of a problem, and then coming to terms with it:

> "After all, showing that a problem is *unsolvable* doesn't appear to be any use if you have to solve it. You need to study this phenomenon for two reasons. First, knowing when a problem is *algorithmically unsolvable* is useful because then you realize that the problem must be simplified or altered before you can find an algorithmic solution. Like any tool, computers have capabilities and limitations that must be appreciated if they are to be used well. The second reason is *cultural*. Even if you deal with problems that clearly are solvable, a glimpse of the unsolvable can stimulate your imagination and help you gain an important perspective on computation."
> Sipser, 1997, p.151, italics added.

This was quintessentially the vision and method adopted by Simon, in framing decision problems to be solved by boundedly rational agents, satisficing in the face of computationally complex search spaces.

The formal decision problem framework for a boundedly rational information processing system can be constructed in one of the following ways: systems of linear Diophantine inequalities, systems of linear equations in non-negative integer variables, integer programming. Solving the former three problems are equivalent in the sense that the method of solving one problem provides a method to solve the other two as well. The Integer Linear Programming (**ILP**) problem and **SAT** can be translated both ways, i.e, one can be transformed into another.

In Velupillai (42), it is demonstrated that **SAT** problem is the meeting ground of Diophantine problem solving and satisficing search, in turn this connection leads to the conclusion that bounded rationality is the superset of orthodox rationality, which had been Simon's point, all along.

Quite apart from the positive aspects of taking a Simonian approach to *Human Problem Solving*, in its analogy with Turing's way of discussing *Solvable and Unsolvable Problems*, the felicity of being able to show that orthodox optimization, by what Simon referred to as the *Olympian* rational agent, is a *special case*[23] of boundedly rational agents, *satisficing* in a computationally underpinned behavioural decision-making context, is, surely, a vinidication of the computable approach to economics!

§5. Beyond Computable Economics

> "Most mathematicians are so deeply immersed in the classical tradition that the idea of abandoning classical logic and mathematics for a constructive approach seems very strange. Those of us interested in approaching mathematics constructively usually need to spend a great deal of time and energy justifying our interest, and we are often dismissed as cranks. …
> ….
> [T]he crux of the insistence by intuitionists (and, presumably, by other strict constructivists as well) that nonconstructive classical mathematics is meaningless is their view that it is not meaningful to talk about completing a computation process requiring an infinite number of steps."

[23] In the same sense in which the reals are 'constructed' from the integers and, then, the rational numbers.

Seldin, 25; p. 105.

Those of us who take a computable approach to economic theory face the same problems, The mind boggles at the thought of one who is not only a computable economist, but who also endorses a constructive and non-standard analytic approach to the mathematisation of economic theory.

What, ‘exactly’, is a computation? The lucid, elementary, answer to this question, entirely in terms of computability theory and Turing Machines, was given by Martin Davis in a masterly exposition almost three and a half decades ago (Davis, 7). Here I am interested in an answer that links the triad of *computation*, *simulation* and *dynamics* in an *epistemological* way. This is because I believe simulation provides the epistemological cement between a computation and the dynamical system that is implemented during a computation – by means of an algorithm, for example.

I shall assume the following *theses*, in the spirit of the *Church-Turing thesis* with which I began this paper, so that I can answer the question 'What is a computation' in an epistemologically meaningful sense.

Thesis 1: Every computation is a dynamical system
Thesis 2: Every simulation of a dynamical system is a computation

Analogously, then: What can we know, what must we do and what can we hope from a computation, which is, by the above claim, a dynamical system? This, in turn, means what can we know, what must we do and what can we hope from studying the behaviour of a dynamical process during a computation? Since, however, not everything can be computed it follows that not every question about a dynamical system can be answered unambiguously. But by the second of the above claims, I have expressed an ‘identity’ between a simulation and a computation, via the intermediation of a dynamical system, which implies that *not everything can be learned* about the behaviour of a dynamical system by simulating it on (even) an ideal device that can compute anything that is theoretically computable (i.e., a Turing Machine, assuming the Church-Turing Thesis). Above all, we cannot delineate, in any meaningful sense - i.e., in an algorithmically decidable sense - between what can be known or learned and that which lies `beyond' this undecidable, indefinable, border, on one side of which we live our scientific lives.

It is humility in the face of this epistemological barrier that one learns to cultivate in approaching the decision problems of economics on the basis of computability theory.

With this as a starting point, the next step beyond Computable Economics is towards Constructive Economics and Intuitionistic Logic – towards an *Intuitionistically Constructive Economics*[24], to untie oneself from the bondage of classical logic, too.

It is, therefore, appropriate to end this exercise with Jeremy Avigad's perceptive, yet pungent, reflection (Avigad, 2009, pp.64-5; italics added):

> "[The] adoption of the infinitary, nonconstructive, set theoretic, algebraic, and structural methods that are characteristic to modern mathematics [….] were controversial, however. At issue was not just whether they are consistent, but, more pointedly, whether they are meaningful and appropriate to mathematics. After all, if one views mathematics as an essentially computational science, then arguments without computational content, whatever their heuristic value, are not properly mathematical. .. [At] the bare minimum, we wish to know that the universal assertions we derive in the system will not be contradicted by our experiences, and the existential predictions will be borne out by calculation. This is exactly what Hilbert's program was designed to do."

And it is precisely in that particular aim Hilbert's Program failed; yet mathematical economics, in all its orthodox modes, adheres to it with unreflective passion.

References

(1). Arrow, Kenneth J (1986), *Letter to Alain Lewis*, July 21, deposited in: Kenneth Arrow Papers, Perkins Library, Duke University.

(2). Avigad, Jeremy (2009), *The Metamathematics of Ergodic Theory*, **Annals of Pure and Applied Logic**, Vol. 157, pp. 64-76.

(3). Brainard , William C & Herbert E. Scarf (2005), *How to Compute Equilibrium Prices in 1891*, **The American Journal of Economics and Sociology:** Special Invited Issue: Celebrating Irving Fisher: The Legacy of a Great Economist , Vol. 64, No. 1, January, pp. 57-83.

(4). Brattka, Vasco (2008), *Plottable Real Number Functions and the Computable Graph Theorem*, **SIAM Journal of Computing**, Vol. 38, No. 1, pp. 303-328.

(5). Bridges, Douglas (1999), *Constructive Methods in Mathematical Economics*, **Journal of Economics**, Supplement 8, pp. 1-21.

[24] Or, alternatively, in terms of *Smooth Infinitesimal Analysis* (**SIA**), and thus an economic theory in mathematical modes that is based on *category theory* and *topos theory*, all three of which have computational underpinnings.

(6). Davis, Martin (1977), **Applied Nonstandard Analysis**, John Wiley and Sons, New York.

(6). Davis, Martin (1978), *What is a Computation?*, pp. 241-267, in: **Mathematics Today - Twelve Informal Essays**, edited by Lynn Arthur Steen, Springer-Verlag, New York.

(8). Davis, Martin, Yuri Matijasevic & Julia Robinson (1986), *Hilbert's Tenth Problem. Diophantine Equations: Positive Aspects of a Negative Solution*, in: **Mathematical Developments Arising from Hilbert Problems**, edited by F.E. Browder, American Mathematical Association, Providence, RI.

(9). Euwe, Max (1929; 2016), *Mengentheoretische Betrachtungen über des Schachspiel* (Communicated by Prof. R. Weitzenböck), translation forthcoming in: **New Mathematics and Natural Computation**, 2016.

(10). Feferman, Solomon (1992), *Preface* to: *Systems of Logic Based on Ordinals*, in: **Collected Wroks of A.M. Turing – Mathematical Logic**, edited by R. O. Gandy & C. E. M Yates, North-Holland, Amsterdam.

(11). Finkelstein, David (1994), *Finite Physics*, pp. 323-347, in: **The Universal Turing Machine – A Half-Century Survey** (Second Edition), edited by Rolf Herken, Springer-Verlag, Wien & New York.

(12). Gandy, Robin (1980), *Church's Thesis and Principles for Mechanisms*, in: **The Kleene Symposium**, edited by J. Barwise, H. J. Keisler and K. Kunen, North-Holland, Amsterdam.

(13). Gandy, Robin (1994), *The Confluence of Ideas in 1936*, pp.51-102, in: **The Universal Turing Machine – A Half-Century Survey** (Second Edition), edited by Rolf Herken, Springer-Verlag, Wien & New York.

(14). Gödel, Kurt (1946,[1965]), Remarks Before the Princeton Bicentennial Conference on Problems in Mathematics, in: The Undecidable - Basic Papers on Undecidable Propositions, Unsolvable Problems and Computable Functions, edited by Martin Davis, pp. 84-88, Raven Press, New York.

(15). Hardy, G. H (1937), *The Indian Mathematician Ramanujan*, **The American Mathematical Monthly**, Vol. 44, No. 3, March, pp. 137-155.

(16). Kleene, Stephen, C (1952), **Introduction to Metamathematics**, North-Holland Publishing Company, Amsterdam.

(17). Kleene, Stephen. C (1981), *Origins of Recursive Function Theory*, **Annals of the History of Computing**, Vol. 3, No. 1, January, pp. 52-67.

(18). Mantel, Rolf, Ricardo (1968), *Toward a Constructive Proof of the Existence of Equilibrium in a Competitive Economy*, **Yale Economic Essays**, Vol. 8, # 1, Spring, pp. 155-196.

(19). Newell, Allen & Herbert. A Simon (1972), **Human Problem Solving**, Prentice-Hall Inc., E8glewood Cliffs, NJ.

(20). Penrose, Roger (1987), **The Emperor's New Mind: Concerning Computers, Minds, and the Laws of Physics**, Oxford University Press, Oxford.

(21). Penrose, Roger (1994), **Shadows of the Mind: A Search for the Missing Science of Consciousness**, Oxford University Press, Oxford.

(22). Péter, Rózsa (1967), **Recursive Functions** (Third Revised Edition), translated from the German by István Földes, Academic Press, New York.

(23). Rabin, Michael O. (1957), *Effective Computability of Winning Strategies*, pp. 147-157, in: **Contributions to the Theory of Games**, Vol. III, edited by M. Dresher, A.W. Tucker and P. Wolfe, Princeton University Press, Princeton, NJ.

(24). Rosser, Barkley. J (1953), **Logic for Mathematicians**, McGraw-Hill Book Company, Inc., New York.

(25). Seldin, Jonathan. P (1981), *A Constructive Approach to Classical Mathematics*, pp. 105-110, in: **Constructive Mathematics** – *Proceedings of the New Mexico State University Conference Held at Las Cruces, New Mexico*, August 11-15, 1980, edited by F. Richman, Springer-Verlag, Berlin & New York.

(26). Simon, Herbert. A (1979), **Models of Thought**, Vol. 1, Yale University Press, New Haven.

(27). Simon, Herbert. A (1996), *Machine as Mind*, Chapter 5, pp. 81-101, in: **Machines and Thought - The Legacy of Alan Turing**, *Volume 1*, edited by Peter Macmillan and Andy Clark, Oxford University Press, Oxford

(28). Sipser. M (1997). *Introduction to the Theory of Computation*, PWS Publishing Company, Boston.

(29). Smale, Stephen (1976), *Dynamics in General Equilibrium Theory*, **American Economic Review**, *Papers and* Proceedings, Vol. 66, No. 2, May, pp. 288-294.

(30). Stewart, Ian, (1992), *Deciding the Undecidable*, **Nature**, Vol. 352, 22 August, pp. 664-5.

(31). Stuart, Andrew & A.R. Humphries (1998), **Dynamical Systems and Numerical Analysis**, Cambridge University Press, Cambridge.

(32). Svozil, Karl (1993), **Randomness & Undecidability in Physics**, World Scientific, Singapore.

(33). Turing, Alan. M (1936-7), *On Computable Numbers with an Application to the Escheidungsproblem*, **Proceedings of the London Mathematical Society**, Series 2, Vol. 42, pp. 230-265.

(34). Turing, Alan. M (1939), *Systems of Logic Based on Ordinals* (**Proceedings of the London Mathematical Society**, Series 2, Vol. 45.

(35). Turing, Alan.M (1948; 1969, 1992), *Intelligent Machinery*, Report, National Physical Laboratory, in: **Machine Intelligence**, 5, pp. 3-23, edited by: B. Meltzer & D. Michie, Edinburgh University Press, Edinburgh; reprinted in: **Mechanical Intelligence - Collected Works *of* Alan Turing**, pp. 107-127, edited by D.C. Ince, North-Holland, Amsterdam.

(36). Turing, Alan. M (1950), *Computing Machinery and Intelligence*, **Mind**, Vol. 50, pp. 433-60

(37). Turing, Alan. M (1952), *The Chemical Basis of Morphogenesis*, **Philosophical Transactions of the Royal Society**, Series B, Vol. 237, pp. 37-72.

(38). Turing, Alan. M (1954), *Solvable and Unsolvable Problems*, **Science News**, #31, pp. 7-23.

(39). Turing, Alan (1986), **A. M. Turing's ACE Report of 1946 and Other Papers**, edited by B. E. Carpenter & R. W. Doran, The MIT Press, Cambridge, Massachusetts.

(40). Uzawa, Hirofumi (1962), *Walras' Existence Theorem and Brouwer's Fixed Point Theorem*, **The Economic Studies Quarterly**, Vol. 8, No. 1, pp. 59-62

(41). Velupillai, Kumaraswamy (2000), **Computable Economics**, Oxford University Press, Oxford.

(42). Velupillai, K. Vela (2010), **Computable Foundations for Economics**, Routledge, London.

(43). Velupillai, K. Vela (2014), ***Constructive*** *and* ***computable*** *Hahn–Banach theorems for the (second) fundamental theorem of welfare economics*, **Journal of Mathematical Economics**, Vol. 54, pp. 34-37.

(44). Velupillai, K. Vela (2016), *Seven Kinds of Computable and Constructive Infelicities*, Forthcoming in: **New Mathematics and Natural Computation, 2016.**

(45). Zurek, W. H (1999), *Algortihmic Randomness, Physical Entropy. Measurement, and the Demon of Chiice*, chapter 5.1, pp. 264-281, in: **Maxwell's Demon 2 - Entropy, Classical and Quantum Information, Computing**, edited by Harvey S. Leff & Andrew F. Rex, Institute of Physics Publishing, Bristol.