AGGREGATE PRODUCTION FUNCTIONS AND NEOCLASSICAL PROPERTIES: AN EMPIRICAL VERIFICATION

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Abstract

Standard postulates concerning the aggregate production function are about marginal productivities - and the associated demand of labor and capital - which are to be negatively related to factor prices, namely the wage rate and the profit rate. The theoretical cases in which these neoclassical properties do not hold are regarded as anomalies. We compute the aggregate values for capital, production and labour and find that the neoclassical postulates do not hold for the whole dataset under consideration.

Keywords: Aggregate Neoclassical Production Function, Cobb-Douglas, CES, Technological Change, Macroeconomics, Growth.

JEL classifications: C61, C63, C67, O47
Introduction

It is a widespread practice among most economists to use the ‘neoclassical’ aggregate production function, especially while constructing macroeconomic models. Routinely used models (for e.g., Solow-Swan, Ramsey-Cass-Koopmans, endogenous growth, overlapping generations, real business cycles, aggregate demand and aggregate supply, dynamic stochastic general equilibrium, computable general equilibrium models, and so on) are all usually based on aggregate CES production functions.

These models often represent an economic system that produces a large number of heterogeneous goods with a few index numbers (one number each for output $Y$, productive capital $K$, total quantity of employed labor $L$, and technological level or knowledge $A$). Samuelson has appropriately referred to this type of aggregate production functions as the surrogate or as-if production function, Samuelson (1962, p.194 fn.1). Such an aggregate representation may be useful, provided that the indexes have certain properties (Fisher, 1922; Frisch, 1936)). However, at the end of the 1960s, it was concluded that aggregation could be problematic.

The problems are of two types. The first are those associated with the technical aggregation from micro-level to macro-level. That is, simple ‘well-behaved’ production functions, after aggregation, do not retain the same functional forms as before aggregation(Fisher, 1969; Shaikh, 1974).

The second are known as value problems, which were addressed during the two Cambridges debate. The conclusion of this debate has been that there exist cases in which the aggregation from a multi-commodity space to a single surrogate production function (see Samuelson (1962)) may result in a production function that is not well-behaved (this problem was admitted by Samuelson (1966) himself). Solow, who acknowledged the problem (Solow, 1976, p.138), observed:

\[ \ldots \text{I have to insist again that anyone who reads my 1955 article [Solow (1955)] will see that I invoke the formal conditions for rigorous aggregation not in the hope that they would be applicable \ldots but rather to suggest the hopelessness of any formal justification of an aggregate production function in capital and labor} \]

Regardless of the widespread acknowledgment that aggregation could be problematic, the (generalized) Cobb-Douglas production function is widely used as an essential element in various models across theories and as a fundamental tool for the empirical assessment of technological progress and productivity growth. There are perhaps two reasons for this. The first is that although one is not assured that the aggregation will always preserve neoclassical properties, there exist, at least in theory, sets of methods for which a neoclassical surrogate production function does in fact exist. The second is due to the fact that the economists questioning the validity of the neoclassical aggregate production function have been unable to convincingly demonstrate the empirical untenability of the production functions of the Cobb-Douglas type (or the generalized CES type). They have also not been able, so it seems, to provide a valid and useful alternative.

\[ ^1 \text{For example let us assume two firms, producing output with the following production functions } Y_1 = F(K_1, L_1) \text{ and } Y_2 = G(K_2, L_2), \text{ respectively. Even if both satisfy the usual neoclassical properties, i.e., positive marginal productivities and positive marginal rates of substitution, it is possible that the function } F + G \text{ may not satisfy the above mentioned neoclassical properties.} \]

\[ ^2 \text{A relevant list of contributions to the two Cambridges debate may be found in Zambelli (2004)} \]
The position taken by Sato (1974, p.383, italics added) is still representative of the state of affairs that is prevalent among the majority of economists today. He argued:

... that there is a not-too-small world in which the neoclassical postulate [i.e. production function] is perfectly valid. So long as we live in that world, we need not to give up the neoclassical postulate [i.e. production function] ... Nonetheless, it is important to realise that there is another world in which the neoclassical postulate may not fare well or is contradicted. An empirical question is which of the two models is more probable.

On the one hand, he admits the existence of the problem, and, on the other hand, he declares a belief that an empirically non-negligible proportion of the world has neoclassical properties. In doing so, he makes the problem of aggregation as a type of curiosum, which is interesting from the theoretical point of view, but irrelevant for empirical applications. This position has not been satisfactorily challenged empirically, demonstrating that the world is or is NOT neoclassical. This empirically unchallenged belief that the world is neoclassical has led to a state of affairs in which productivity measurements (total and multiple factor productivities) and measurements of technological progress and economic efficiency in general are all based on the aggregate neoclassical production function. In fact, the points of departure for these measures are still the works of Solow (1957), Farrell (1957).

Due to the discovery of a powerful algorithm, the FVZ-algorithm (Zambelli et al., 2014, Sec. 4), and the availability of a rather complete data set (Timmer, 2012, WIOD), here we are able to compute surrogate production functions for a wide range of data belonging to 30 countries and for 15 years.

We examine whether these actual production functions possess neoclassical properties. The results presented here are, in our view, rather conclusive: surrogate production functions, contrary to what generally believed or postulated by faith, do not have neoclassical properties.

1 Aggregate Neoclassical Production Function - A Short Review

Economists often assume that the production of an economic system can be described with an as-if production function. The neoclassical aggregate production function is a mathematical relation that links the output with the inputs and which holds specific properties (Solow, 1955, 1956, 1957; Arrow et al., 1961; Ferguson, 1969; Shephard, 1970). We consider the simple case with one output $Y$ and two physical inputs, $K$, $L$.

$$Y = F(K, L)$$ (1.1)

There are three basic sets of assumptions that are often considered to be necessary for the above functional form to be neoclassical.

First Set of Assumptions. Law of positive, but decreasing marginal productivities. These assumptions are those of convexity, continuity and differentiability. This translates to the following properties for the function $F$: $\partial Y/\partial K = F_K(K, L) > 0$; $\partial^2 Y/\partial K^2 = F_{KK}(K, L) < 0$; $\partial Y/\partial L = F_L(K, L) > 0$, $\partial^2 Y/\partial L^2 = F_{LL}(K, L) < 0$;
These, in turn, are characterized as the law of positive, but decreasing marginal productivities.

**Second Set of Assumptions.** *Theory of Social Distribution based on Marginal Productivities.* It is assumed that markets operate in such a way that the wage rate $w$ is equal to the marginal productivity of labor and the interest rate (or rental cost of capital) $r$ is equal to the marginal productivity of capital. The representative producer is assumed to maximize profits, given the constraints and that the competition among producers would lead him to choose the production level associated with the minimization of (factor) costs. Given the profit function

$$
\Pi = pY - rK - wL
$$

(1.2)

the first order conditions, assuming that the producer is a *price-taker* and that prices are assumed to be fixed\(^3\), have to be fulfilled in order to maximize the profits. The first order conditions for the maximization of profits are given by:

$$
\frac{\partial \Pi}{\partial K} = p\frac{\partial Y}{\partial K} - r = 0 \implies r = pF_K = p\frac{\partial Y}{\partial K}
$$

(1.3)

$$
\frac{\partial \Pi}{\partial L} = p\frac{\partial Y}{\partial L} - w = 0 \implies w = pF_L = p\frac{\partial Y}{\partial L}
$$

(1.4)

Equation 1.3 is the *demand schedule for capital* and eq. 1.4 is the *demand schedule for labor*. The physical world of production and that of the exchange maybe linked considering the Marginal Rate of Technical Substitution (MRTS), which is the change of one factor that is necessary to accommodate a change of another factor, so as to keep the production along the same isoquant. We have the following relation:

$$
0 = dY = \frac{\partial Y}{\partial K} dK + \frac{\partial Y}{\partial L} dL \Rightarrow MRTS = -\frac{dK}{dL} = \frac{\partial Y}{\partial L} / \frac{\partial Y}{\partial K}
$$

(1.5)

Substituting eq. 1.3 and eq. 1.4 into 1.5 we can link a technical relation with factor prices:

$$
MRTS = -\frac{dK}{dL} = \frac{w}{r}
$$

(1.6)

**Third Set of Assumptions.** *Homogeneity of degree 1 and Constant Elasticity of Substitution - CES.* Arrow et al. (1961) introduce additional feature, which include homogeneity of degree 1 - i.e. $AF(\lambda K, \lambda L) = \lambda AF(K, L) = \lambda Y$ - and Constant Elasticity of Substitution. The elasticity of substitution is given by:

$$
\sigma = -\frac{dK/K}{dL-L} \frac{dMRTS}{MRTS} = \frac{\partial \ln(K/L)}{\partial \ln(MRTS)} = \frac{\partial \ln(K/L)}{\partial \ln(w/r)}
$$

(1.7)

Clearly, with $\sigma = 1$, an increase in the capital-labor ratio will be matched by an

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\(^3\)A standard assumption is to assume that the prices $p$, $w$ and $r$ are independent of the quantities. This is an assumption which is not justified by actual operation of markets: clearly these prices are not independent of the quantities demanded or supplied. To assume the constancy of prices means that $\partial p/\partial L = \partial p/\partial K = \partial r/\partial L = \partial r/\partial K = \partial w/\partial L = \partial w/\partial K = 0$
exact increase in the wage-profit ratio. This is the case in which although it is possible to observe an increase in the capital-labour ratio, this will be associated with constant shares \( \frac{wL}{pY} \) and \( \frac{rK}{pY} \).

There are different functional forms that would be consistent with the above assumptions. A widely adopted functional form is the Cobb-Douglas production function, which is a special case of the generalized CES-production function:\(^4\)

\[ Y = F(K, L) = AK^\alpha L^{1-\alpha} \]  

(1.8)

The isoquant associated with the above Cobb-Douglas production function is:

\[ K = \left( \frac{\bar{Y}}{AL^{1-\alpha}} \right)^{1/\alpha} \]  

(1.9)

where \( \bar{Y} \) is a given constant level of output. The associated wage-profit curve is derived by substituting the marginal productivities associated with 1.8 into eqs. 1.3 and 1.4:

\[ w = p(1 - \alpha) \left( \frac{p\alpha}{r} \right)^{\alpha/(1 - \alpha)} \]  

(1.10)

It is interesting to note that even though the position of the isoquant, eq. 1.9, is obviously dependent on the level of activity, the wage-profit curve does not depend on it. The standard procedure is to assume the above and to estimate the parameters of the chosen production function. Here, for the sake of simplicity and exposition, we assume a production function as the one above, eq. 1.8. The parameters \( \mathcal{A} \) and \( \alpha \) are estimated from the actual data. Figure 1.1 reports the values of capital-labor ratio, \( K/L \) of the aggregate data relative to the year 2009 and of the output-per-labor ratio, \( Y/L \) of 30 countries\(^5\). The requirement that the efficient production set should be convex leads one to restrict the number of efficient points that need to be considered for the estimation of \( \mathcal{A} \) and \( \alpha \). There are different methods for estimating the best fit production function. The important point here is to realize that these different methods (see Kao et al. (2014)) would be estimations that would be near the thick seemingly smooth line.

Today it is standard procedure to proceed as if the CES type aggregate production functions can represent the production function of a national production system.

2 Samuelson’s Surrogate Production Function

A very important and a natural question is to ask whether it is sound to assume that a production system, where there are many products and many different methods to

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\(^4\)Arrow et al. (1961) have suggested the following generalized CES-Production function \( Y = F(K, L) = \gamma_1[K^\rho + \gamma_2L^\rho]^{1/\rho} \) where \( \rho = (\sigma - 1)/\sigma \).

\(^5\)The countries considered are: (AUS) Australia; (FIN) Finland; (KOR) Korea; (AUT) Austria; (FRA) France; (MEX) Mexico; (BEL) Belgium; (GBR) Great Britain; (NLD) Netherlands; (BRA) Brazil; (GRC) Greece; (POL) Poland; (CAN) Canada; (HUN) Hungary; (PRT) Portugal; (CHN) China; (IDN) Indonesia; (RUS) Russia; (CZE) Czech Republic; (IND) India; (SWE) Sweden; (DEU) Germany; (IRL) Ireland; (TUR) Turkey; (DNK) Denmark; (ITA) Italy; (TWN) Taiwan; (ESP) Spain; (JPN) Japan; (USA) United States.
produce them, can be represented with a simple system like the Cobb-Douglas. We know that the economic system is producing a great variety of commodities.

Samuelson (1962) proposed a method that is meant to provide a theoretical justification for the simplification, known as the aggregation.

One need never speak of the *the* production function, but rather should speak of a great number of separate production functions, which correspond to each activity and which need have no smooth substitutability properties. All the technology of the economy could be summarized in a whole book of such production functions, each page giving the blueprint for a particular activity. Technological change can be handled easily by adding new options and blue prints to the book ... Finally it is enough to assume that there is but one ‘primary’ or non-producible factor of production, which we might as well call labor ... All other inputs and outputs are producible by the technologies specified in the blueprints Samuelson (1962, p.194)

The book of blueprints can be seen as different entries of the input-output tables. We observe from the actual tables that $b_i$ units of commodity $i$ can be produced with $s_i$ different alternative methods\(^6\).

$$\phi(z_i; i) : a_{1i}, a_{2i}, \ldots, a_{si}, \ell_i \rightarrow b_i$$  \hspace{1cm} (2.1)

\(^{6}\)The notation here is slightly different from standard mathematical notation. This is a notation familiar to the users of Matlab for multiple dimension arrays. The numbers inside parenthesis identify the dimension, i.e. rows, columns, 3\(^{rd}\)-dimension, 4\(^{th}\)-dimension and so on. The symbol : stands for all the numbers in that dimension, and 1 : $s$ means from 1 to $s$ and so on. $\phi(z_i; i)$ identifies an entry for the multiple dimension array $\phi$, where $z_i$ identifies the row, : means for all columns and $i$ the third dimension.
where: $i = 1, \ldots, n$; $j = 1, \ldots, n$; $z_i = 1, \ldots, s_i$. $a_{ij}^z$ is the input of commodity $j$ in producing a good $i$ using a method $z_i$. $s_i$ is the number of available methods for producing the good $i$ and $n$ is the number of goods.

The set of methods for producing good $i$ – i.e., the set of blueprints for the production of $i$ – can be represented in matrix notation as $\Phi(1 : s_i, 1 : (n + 2), i)$. The set of all the available methods is given by the following set of activities $\Phi = \{ \Phi(:,:,1) \cup \Phi(:,:,2) \ldots, \Phi(:,:,n) \}$ (see Zambelli et al. (2014)).

The cardinality of the above set of methods can be very large and subsets of the above methods can exhibit, in principle, a great variety of mathematical properties. Consequently, whether a production system has, for example, the convexity property and hence does approximate a neoclassical production function depends on the ‘actual’ structure of $\Phi$ (see Zambelli et al. (2014)). The set of all available methods is given by the following set of activities $\Phi = \{ \Phi(:,:,1) \cup \Phi(:,:,2) \ldots, \Phi(:,:,n) \}$

**Figure 2.1: Cobb-Douglas: year 2009.** The graphs above are based on the estimate of the Cobb-Douglas where the data is from fig. 1.1. The estimated value for $A$ is 34.6 and for $\alpha$ is 0.224. The north-west graph, Cobb-Douglas wage-profit curve, is computed with eq. 1.10. The north-east graph, The Demand for Labor curve, is computed with eq. 1.4 and the south-west graph represents the Demand for Capital curve derived from eq. 1.3. The south-east graph denotes the isouquant curve derived from eq. 1.9.

Hence, a $n$-commodity output vector can be generated by using each combination of methods, which belongs to set $\Phi$. There are a total $s = \prod_{i=1}^{n} s_i$ of these combinations. Given one of these combinations, $z = [z_1, z_2, \ldots, z_n]'$, we have one production possibility. The heterogeneous production of a system would then depend on the level of employment and the methods of production adopted. The quadruple ($A^z, L^z, B^z, x$) is the standard representation of an input-output system where $A^z$ is the set of the inputs used, $L^z$ is the vector of the amount of labor that is necessary and $B^z$ is the associated output. $x$ is
the vector defining the level of activity $^7$.

For a chosen system, $z$, (i.e., a triple $B^z, A^z, L^z$) the production prices that would assure accounting equilibrium are those that allow the following relation to hold:

$$A^z(1 + r)p + L^zw = B^zp$$  \hspace{1cm} (2.2)

For a given uniform profit rate $r$ and uniform wage rate $w$, there exists a price vector $p$ that would allow the system to remain productive for the subsequent periods as well:

$$p^z(r, w) = [B^z - A^z(1 + r)]^{-1}L^zw$$  \hspace{1cm} (2.3)

We then choose a numéraire, a vector composed of different proportions of the $n$ produced goods forming the input-output tables,

$$\eta^z(r, w) = 1$$  \hspace{1cm} (2.4)

we are now in a position to define the wage-profit curve. By substituting 2.3 into 2.4, we obtain the wage-profit curve associated with the set of methods $z$:

$$w^z(r, \eta) = \eta[z[B^z - A^z(1 + r)]^{-1}L^z]^{-1}$$  \hspace{1cm} (2.5)

This is the wage-profit curve associated with system $z$. Substituting 2.5 into 2.3 we obtain the price vector

$$p^z(r, \eta) = [B^z - A^z(1 + r)]^{-1}L^z[\eta[z[B^z - A^z(1 + r)]^{-1}L^z]^{-1}$$  \hspace{1cm} (2.6)

The price vector $p^z(r, \eta)$ is a function of the particular set of methods $z$, the profit rate $r$ and the numéraire.

Samuelson (1962) proposed to simplify the theory of production not by way of assumption, but by construction. The straight lines in the north-west graph of Fig. 2.1 each represent a different set of methods $z_i$, which in turn represent a wage-profit relation $^8$. The north-west graph is qualitatively equivalent to Figures 1 and 2 present in Samuelson (1962, p.195, p.197). It can be seen that there is an envelope that would be formed as the outer frontier of a large number of straight lines, i.e. a large number of different set of methods $z_i$. He did originally assume that the wage-profit curves were straight lines. This assumption was subsequently challenged during the Cambridge Capital Controversy (see Garegnani (1966); Pasinetti (1966); Bruno et al. (1966)). He had to give up this assumption during the debate that followed (Samuelson, 1966).

\hspace{1cm} $^7A^z = \Phi(z, 1 : n, 1 : n) = \begin{bmatrix} a_{11}^1 & a_{12}^1 & \cdots & a_{1n}^1 \\ a_{21}^1 & a_{22}^1 & \cdots & a_{2n}^1 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^1 & a_{n2}^1 & \cdots & a_{nn}^1 \end{bmatrix}$;  
$L^z = \Phi(z, n + 1, 1 : n) = \begin{bmatrix} \ell_{11}^z \\ \ell_{21}^z \\ \vdots \\ \ell_{n1}^z \end{bmatrix}$;  
$B^z = \text{diag}(\Phi(z, n + 2, 1 : n)) = \begin{bmatrix} b_{11}^z & 0 & \cdots & 0 \\ 0 & b_{22}^z & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn}^z \end{bmatrix}$;

\hspace{1cm} $^8$For a more detailed explanation see also Pasinetti (1977, Ch. 6, fn. xxxx).
The surrogate or as-if production function is derived from the level of employment and the value of capital that is associated with the wage-profit curves that form the envelope.

The value of aggregate net national product associated with a given set of methods, \( z \), is given by:

\[
Y^{z}_{\text{val}}(r, x, \eta) = x'[B^z - A^z]p^z(r, \eta)
\]  

where \( x \) is the level of activity vector\(^9\).

The value of aggregate capital per worker is given by:

\[
K^{z}_{\text{val}}(r, x, \eta) = x'A^z p^z(r, \eta)
\]  

The net national product per unit of labor associated with a given set of methods, \( z \), is given by:

\[
y^{z}_{\text{val}}(r, x, \eta) = \frac{Y^{z}_{\text{val}}(r, x, \eta)}{x'L^z} = \frac{x'[B^z - A^z]p^z(r, \eta)}{x'L^z}
\]  

The value of capital per worker is given by:

\[
k^{z}_{\text{val}}(r, x, \eta) = \frac{K^{z}_{\text{val}}(r, x, \eta)}{x'L^z} = \frac{x'A^z p^z(r, \eta)}{x'L^z}
\]

Different activity levels \( x \) are associated with different produced and distributed quantities and hence different consumption possibilities. The same value of output, \( y^{z}_{\text{val}}(r, x, \eta) \), or of capital, \( k^{z}_{\text{val}}(r, x, \eta) \), may be associated with an enormous number of vectors \( x^{10} \). We shall consider the activity level for which the value of aggregate net national product is the highest.

For each triple \((B^z, A^z, L^z)\), numéraire, \( \eta \), and profit rate \( r \), there is an efficient activity vector, \( x^* \) which is determined using linear programming algorithm.

\[
\text{max} \quad x'[B^z - A^z]p^z(r, \eta)
\]

s.t.

\[
x'[B^z - A^z] \geq 0'
\]

\[
x'L^z - e'L^z = 0'
\]

\[
x' \geq 0'
\]

The highest values of net national product per labor and capital per labor product are given by:

\[
y^{z}_{\text{val}}(r, x^*, \eta)
\]

\[
k^{z}_{\text{val}}(r, x^*, \eta)
\]

\(^9\)It is important to recall that the production prices and the wage rate are not dependent on the activity level. This result is known as the non-substitution theorem. On the origins of the non-substitution theorem, see Arrow (1951); Koopmans (1951); Samuelson (1951). A more recent treatment is given in Mas-Colell et al. (1995), pp.159-60. Also see, Zambelli (2004, footnote 2, p.105))

\(^{10}\)This is a very serious problem. When we search for whether the system exhibits neoclassical properties, it would also be important to study the allocations implied by this. In the neoclassical literature, this problem is not addressed by assuming that the output and capital, and hence consumption, are all homogeneous. Obviously, this cannot be done for the case of heterogeneous production.
The outer envelope of all $m$ possible wage-profit curves is the wage-profit frontier:

$$w_{WPF}^E(r, \eta) = \max \left\{ w^{x_1}(r, \eta), w^{x_2}(r, \eta), \ldots, w^{x_m}(r, \eta) \right\}$$

(2.17)

where $E$ is a subset of $\Phi$, $(E \subset \Phi)$.

The domain of $w_{WPF}^E(r, \eta)$ is composed of $v$ intervals. The junction between the different intervals are called switch points - points where the dominance of one wage-profit curve is replaced by another.

$$r \in \left[0, \hat{r}_1 \cup [\hat{r}_1, \hat{r}_2] \cup \ldots \cup [\hat{r}_{v-2}, \hat{r}_{v-1}] \cup [\hat{r}_{v-1}, R_{WPF}^E] \right]$$

(2.18)

where $\hat{r}_k$ ($k = 1, 2, \ldots, v - 1$) are the switch points and $R_{WPF}^E$ is the maximum rate of profit of $w_{WPF}^E(r, \eta)$. These intervals are relatively few with respect to the very large number of possible combination of methods.

Each interval, $k$, is the domain of a wage-profit curve that was generated by the set of methods $z_{(k)}$. The whole set of methods that contribute to $w_{WPF}^E(r, \eta)$ may be represented as a matrix:

$$Z_{WPF}^E = [z^{(1)}, z^{(2)}, \ldots, z^{(k)}, \ldots, z^{(v)}] = \begin{bmatrix} z^{(1)}_1 & z^{(1)}_2 & \ldots & z^{(1)}_v \\ z^{(2)}_1 & z^{(2)}_2 & \ldots & z^{(2)}_v \\ \vdots & \vdots & \ddots & \vdots \\ z^{(k)}_1 & z^{(k)}_2 & \ldots & z^{(k)}_v \\ \vdots & \vdots & \ddots & \vdots \\ z^{(v)}_1 & z^{(v)}_2 & \ldots & z^{(v)}_v \end{bmatrix}$$

(2.19)

We have now all the elements for defining Samuelson’s Surrogate Production Function. The suffix $WPF$ indicates values associated with the outer envelope defined by $Z_{WPF}^E$, eq. 2.19. To sum up the Surrogate Production Function is given by:

$$\text{aggregate output per worker: } y_{WPF}^{val}(r, x_{iso}^+(r), \eta)$$

(2.20)

$$\text{aggregate capital per worker: } k_{WPF}^{val}(r, x_{iso}^+(r), \eta)$$

(2.21)

where $x_{iso}^+(r)$ is the activity level that minimizes the value of capital necessary for the production of the isoproduct value $Y_{iso}^{WPF}(r, x_{iso}^+(r), \eta)$ (see the linear programming eqs. 2.11).

The Surrogate Isoquant would be given by the values that would generate the same value of the output:

$$\text{Isoquant - aggregate output: } Y_{iso}^{WPF}(r, x_{iso}^+(r), \eta) = x_{iso}^+ [B_{WPF} - A_{WPF}] p_{WPF}(r, \eta)$$

(2.22)

$$\text{Isoquant - aggregate capital: } K_{iso}^{WPF}(r, x_{iso}^+(r), \eta) = x_{iso}^+ A_{WPF} p_{WPF}(r, \eta)$$

(2.23)

$$\text{Isoquant - aggregate labor: } L_{iso}^{WPF}(r, x_{iso}^+(r), \eta) = x_{iso}^+ L_{WPF}$$

(2.24)

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11The number of possible curves is enormous. In the database that we use, there are 31 sectors and 30 countries. This means that in order to determine the yearly wage-profit frontier, we need to compute $31^{30}(\approx 5.5 \times 10^{44})$ wage-profit curves. Either one computes all these curves first or one should use an algorithm that reduces the computational time. We use one such algorithm, see Zambelli et al. (2014).
3 Empirical Verification

3.1 Data and the Choice of the Numéraire(s)

We use data from the World Input-Output Database Timmer (2012) which is publicly available and it provides detailed input-output data at the industrial level for 35 industries from 1995-2011. The data set is composed of national input-output tables of 40 countries that includes 27 EU countries and 13 other major industrial countries. For more details regarding the construction of Input-Output tables in WIOD database, see Dietzenbacher(2013).

In this article we have confined our analysis to a subset of 30 countries. Furthermore, we have reduced the total sectors or industries to 31. We are considering only those industries that belong to the core of the ‘production’ system. The National Input-Output tables (NIOT) have been adjusted so as to include the imports of means of production. Hence, the methods associated with each sector would be the inputs of internally produced goods plus the inputs of the imported goods. All current period values have been appropriately adjusted using relevant price indexes. For this, we have used the data on price series that are available in the Social and Economic Accounts (SEA) section of the WIOD database (Timmer (2012)). The unique aspect of SEA is that it offers data at the industry level.

Once the above adjustments have been made, we organize the means of production, labour inputs and the gross output as in the multi-dimensional matrix. This enables us to enumerate all the possible combinations of methods of production with the vectors $z$ and associate them to production systems formed by the triple: $A^z$, $L^z$, $B^z$. We have used this information to compute the yearly wage-profit frontier and the global inter–temporal frontier (and all the methods used at the frontier, $Z_{WPF}^E$ as well).

3.2 An example: global frontier for 2009.

The knowledge of $Z_{WPF}^E$ allows for the computation of the wage-profit frontier. For example, the outer envelope of all the wage-profit curves for the year 2009 that is reproduced in the north-west graph of Fig. 3.1 is computed based on the 64 curves that dominate all the other $31^{30}(\approx 5.5 \times 10^{41})$ curves in 2009, $Z_{WPF}^E_{2009}$, where $E_{2009} \subset \Phi$.

Samuelson assumed that $y_{val}^{WPF}(r, x^*_{iso}(r), \eta)$, (eq. 2.20), $k_{val}^{WPF}(r, x^*_{iso}(r), \eta)$ (eq. 2.21), would have neoclassical properties.

We have computed the values of the system (eqs. 2.20–2.21) from empirically observed data. We would like to reiterate that we are able to perform these computations because, for the first time, we have been able to extract information regarding the methods present at the frontier, $Z_{WPF}^E_{2009}$.

Fig. 3.1 reports some of the computations relative to 2009. The south-west graph of Fig. 3.1 is the isozoom, which is computed following the as-if procedure in Samuelson.

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13 We are now in the position to compute the isozoom associated with the wage-profit frontier, which is derived from heterogeneous production.
Figure 3.1: Year 2009: Aggregate Values and Heterogeneous Production. *The north-west graph is the wage-profit frontier, which is the envelope of the wage-profit curves. The north-east graph, The Demand for Labor at Isoquant is the quantity of labor necessary for the production of the same value quantity of the output, eq. 1.4. The south-west graph is the Demand for Capital curve derived from eq. 1.3. The south-east graph is isoquant curve derived from eq. 1.9*

Figure 3.2: Year 2009. *Convexification. Zoom-in version of Fig. 3.1. The thick lines are consistent with the neoclassical postulates. The dashed lines show the convexification.*
son (1962). Clearly, the isoquant fails to be consistent with the neoclassical set of assumptions.

The wage-profit frontier (north-west graph) and the Demand for Labor at Isoquant (north-east graph) are negatively sloped and this feature is independent of whether the set of methods $\Phi$ has neoclassical properties. These features would apply to any set of methods. It is well known and proven that the wage-profit frontier would always be negatively sloped (see Sraffa (1960); Samuelson (1962)). Also, the Demand for Labor at Isoquant is negatively sloped with respect to the wage rate. But, this is to be expected: as the wage rate decreases, the most efficient methods of production might be those that utilize more labor. The neoclassical requirement is that as more labor is employed along the isoquant, less capital should be employed. An inspection of the south-west graph of fig. 3.1 makes it clear that this is in fact not the case. At least, it is not the case for the whole domain.

At first glance, a scholar trained to think in terms of marginal productivities and substitution among factors may find the positive relation in the isoquants between labor and capital as disturbing and counter intuitive. Although this may be an unpleasant result, it is an actual possibility and in fact, as we will see, it is the normal case. As the profit and wage rates change, there is a change in accounting equilibrium prices. Eventually and/or consequently, there would also be a change in the most efficient methods of production.

The usual temptation is to ‘fix’ this by convexification. Figure 3.2, which is enlarged with respect to fig. 3.1, shows the convexified isoquant. The thick lines would be consistent with respect to the neoclassical assumptions, but the envelope would not be convex. It is the dotted lines, which exclude most of the thick lines from the mapping, that would be coherent with the neoclassical assumptions. This might appear to be a reconciliation with standard neoclassical approach, but it is incorrect. To convexify would mean excluding highly efficient solutions from the feasible, efficient production possibility frontier.

If the figures were relative to a homogeneous production, an output produced using relatively higher quantity of physical inputs instead of using lower inputs, production would be considered inefficient. If we lived in the homogeneous-one-good-world, this assessment would be correct. But it is worth remembering that the plotted values correspond to a system of heterogeneous production, where each point corresponds to a different activity level. More importantly, they correspond to a different set of methods and hence imply that a different net national product to be produced and distributed.

In Fig.3.3, we see the gains in the value of NNP per worker if we assume the most efficient methods (keeping the employment level, sectoral distribution and Capital-Labor ratio as fixed as per 2009 levels). We can see that all the countries, including those that are considered to be the most efficient (after convexification as in Fig. 1.1) according to standard methods, would be able to produce at a much higher level. Therefore, to convexify here would mean forcing the systems to produce at highly sub-efficient levels.

Now we turn to whether these results relative to 2009, can be extended to the whole period, i.e., from 1995 to 2009. Furthermore, it is also important to assess the sensitivity of the results with respect to some of the elements employed for our investigation.
Figure 3.3: Output-Labor Ratio versus Capital-Labor Ratio - 2009, Data from Timmer (2012). The lowest line shows the actual historical values as in Fig. 1.1. The highest lines are the values associated with efficient production methods, computed by keeping labor distribution among the sectors same as that of the historical observation and having the same capital-labor ratio. The values for aggregate Capital and Output are in terms prices with 1995 as the base year, as in WIOD Tables

3.3 Results and their Robustness

We can now attempt to verify whether the Surrogate Production Function has neoclassical properties. We have computed the values of the Surrogate Production Function, as-if CES or as-if Cobb-Douglas relative to the period going from 1995 to 2009, with a set of different numeraires. We have the following results.

1. Neoclassical properties of the isoquants: Neoclassical properties require that

\[
\frac{\Delta K_{WPF}^{\text{iso}}(r, x_{\text{iso}}^*, \eta)}{\Delta L_{WPF}^{\text{iso}}(r, x_{\text{iso}}^*, \eta)} \leq 0
\]  

(a) Global Result \( \forall r \in [0, R_E^{WPF}] \). A single numéraire, for which the above condition holds for the entire domain has not been found.

(b) Partial Result. We compute the ratio between the sum of the intervals of the domain \( r \) in which the above relation holds (numerator) and the total domain \( (r \in [0, R_E^{WPF}], \text{denominator}) \). The above relationship holds, on average (of the numeraires) only for 6.95\% (standard deviation 0.79\%) of the entire domain of the isoquant.

14 The set of numeraires is composed of standard commodities (see Sraffa (1960, pp.18-25)) relative to all the wage-profit curves that contribute to the fifteen yearly wage-profit frontiers. For the year 1995, the frontier is composed of 79 curves and consequently, 79 associated standard commodities, i.e. 79 numeraires. Similarly, there are 74 curves for 1996, 59 for 1997 and so on. For 15 years, we a total of 1004 standard commodities, hence a total of 1004 numeraires. For each standard commodity, we have computed the values, eqs. 2.20–2.21 and eqs. 2.22–2.24, for each year. We have 15060 instances (1004 x 15 years)
Comment: This is an important result that demonstrates that the Surrogate Production Function is not neoclassical.

2. Capital-Labor Ratio. Neoclassical Properties require that:

$$\Delta k_{val}^{WPF}(r, x^{*}(r), \eta) \leq 0$$

(a) Global Result $\forall r \in [0, R_{E}^{WPF}]$. The Capital-Labor ratio is negatively sloped for the whole domain, for $4.46\%$ of the instances (671 times over 15060).

(b) Local Result. We compute the ratio between the sum of the intervals of the domain $r$ where the above relation holds (numerator) and the total domain ($r \in [0, R_{E}^{WPF}]$ - denominator). The above relationship holds on average for $80\%$ (standard deviation $13.7\%$) of the entire domain.

Comment: Clearly, as the profit rate increases the wage rate decreases. This means that the cost of labor would tend to decrease. Hence, for most sectors, it might become more convenient to shift towards methods of production where less capital in value is used. Nevertheless, this does not imply anything with respect to the cost of the capital used, which would in turn depend on the new accounting equilibrium production prices, which change as well. It is still possible that the production prices are such that as total labor increases, capital increases as well. We observe that $\Delta k_{val}^{WPF}(r, x^{*}_{iso}(r), \eta) \leq 0$ is true only for $6.95\%$ of the cases.

3. Output-Capital Ratio. Neoclassical properties require that:

$$\Delta \left( \frac{y_{val}^{WPF}(r, x^{*}(r), \eta)}{k_{val}^{WPF}(r, x^{*}(r), \eta)} \right) \geq 0$$

(a) Global Result $\forall r \in [0, R_{E}^{WPF}]$. The Output-Capital Ratio is never positively sloped for the whole domain.

(b) Local Result. We compute the ratio between the sum of the intervals of the domain $r$ where the above relation holds and the total domain ($r \in [0, R_{E}^{WPF}]$). The required relationships hold in $1.4\%$ (standard deviation $0.79\%$) of the entire domain, on an average.

Comment: This result is very surprising. As the profit rate increases, the wage rate decreases and hence the cost of the labor used decreases. There is a substitution in the methods of production. In the Cambridge Capital Controversy, it is a known possibility that as the profit rate increases, the output-capital ratio may decrease, but in the normal case, it increases. But here it increases for only for $1.4\%$ of the total intervals. We would like to remind the reader that the departure of this analysis from the rest is the use of actual data and that we have computed the frontier that comprises the most efficient methods.

4. Price Monotonicity. The neoclassical principle requires that the sectoral prices change monotonically as the profit rates change. We find that:
(a) **Global Result** \(\forall r \in [0, R_e^{WPF}]\). There is no instance where the prices of all the sectors are monotonic functions of the profit rate.

(b) **Local Result.** The average of the percentage of the sectors for which the prices behave monotonically is about 14\% (which is equivalent to 4-5 sectors on a total of 31 sectors, where prices behave monotonically). The standard deviation is 6\% (which is equivalent circa to one sector).

![Figure 3.4: Normalized Sectorial Prices, Year 2009. The sectoral prices have been computed as in 2.20. Here they are normalized. The numéraire is the standard commodity relative to the first wage-profit curve of the 2009 wage-profit frontier.](image)

**Comment:** The prices here are those associated with the wage-profit frontier. The fact that they do not behave monotonically is contrary to the notion of capital intensive or labor intensive methods, which should be a necessary condition for a set of methods to be characterized as being neoclassical. As an example we report a selection of sectoral prices (normalized to 1 for \(r = 0\)) relative to the year 2009 wage-profit frontier. For the chosen year, only two sectors have price functions that are monotonic and the remaining 29 of them are clearly non-monotonic. Fig. 3.4 shows only 13 of these 29 functions, i.e. those that are not only non-monotonic, but also cross the average of all the prices: neoclassical postulates would require that they should not.

5. **Elasticity of Substitution.** We have computed the elasticity of substitution for each year.

\[
\sigma_{WPF}^{\eta}(r, \eta) = \frac{\Delta \ln (k_\text{cal}(r, \mathbf{x}^\ast(r), \eta))}{\Delta \ln \left( \frac{w_{WPF}(r, \eta)}{r} \right)}
\]

(3.4)

For each set of methods associated to the yearly wage-profit frontiers and numéraire.

(a) **Result.** Here we present the results as an average across profit rates and for all the instances. The average value of the elasticities is 0.38 (standard deviation 2.33).
Comment: The average of the standard deviations is very far from 1. Furthermore, the high standard deviation indicates that it is very far from being constant and does indicate high variation in the elasticity of substitution.

4 Aggregate production functions are NOT neoclassical

Solow (1962), starts his article titled ‘Substitution and fixed proportions in the theory of capital’ saying:

I have long since abandoned the illusion that participants in this debate actually communicate with each other.

But communication of one sort or the other has been always going on. Samuelson (1962) has tried to set the foundations so that a system of heterogeneous production could be represented as-if it were a homogeneous production. Samuelson’s Surrogate Production Function was challenged during the Cambridge Capital Controversy in the 60s. The special issue of the Quarterly Journal of Economics, also known as the ‘QJE Symposium’, was devoted entirely to this debate. On that occasion, Samuelson admitted that some problems do exist:

Pathology illuminates healthy physiology. Pasinetti, Morishima, Bruno-Burmeister-Sheshinski, Garegnani merit our gratitude for demonstrating that reswitching is a logical possibility in any technology, indecomposable or decomposable ... There often turns out to be no unambiguous way of characterizing different processes as more “capital-intensive,” more “mechanized,” more “roundabout” ... If all of this causes headaches for those nostalgic for the old time parables of neoclassical writing, we must remind ourselves that scholars are not born to leave an easy existence. We must respect, and appraise, the facts of life (Samuelson, 1966, p.582-3).

What was recognized as a pathology was the possibility that for some regions in the domain of the profit rate $r$, the change in the value of the capital-labor ratio $\Delta^WPF_{val} (r, x^*(r), \eta)$ could be positive, and not negative as required by neoclassical theory.15

Like Samuelson, there were others who did admit to the existence of this problem, but it was considered to be a pathology, or a perversity or a paradox. Lacking empirical evidence, there has been a general tendency to declare a sort of faith regarding the tenability of the neoclassical cases (see on this Carter (2011)). At the end of the 1960s Ferguson wrote:

[The] validity [of the Cambridge Criticism of neoclassical theory] is unquestionable, but its importance is an empirical or an econometric matter that

15This is also known as capital reversing or the Wicksell effect. One special case of capital reversing is that of reswitching, i.e. a case where the methods of production that were efficient for high profit rates would become efficient again for low profit rates. In the case of reswitching this feature is independent of the chosen numéraire $\eta$. It is a case which is indisputably non-neoclassical. Unfortunately, the attention of many authors has been on reswitching and not on capital reversing. For the 15 yearly wage-profit frontier that we have computed, which envelope more than 50 wage-profit curves, we did not find a single instance of reswitching. However, as we have seen, capital reversing is extremely frequent.
depends upon the amount of substitutability there is in the system. Until econometricians have the answer for us placing reliance upon neoclassical economic theory is a matter of faith. I personally have the faith (Ferguson, 1969, p. xv; emphasis added).

It so happened that there were also others that declared their faith as well. Surprisingly, econometricians or economists have never really delivered a satisfactory answer. Macroeconomic theory has adopted the Robinson Crusoe type models, where capital is homogeneous with output and the Cobb-Douglas or CES type production functions are assumed to hold. This has no empirical justification: aggregation problems are ruled out by assumption.

The highly questionable practice of assuming Cobb-Douglas like production functions a priori has not been abandoned even when studies have seriously disputed the statistical validity of the empirical estimations of the aggregated functions (Simon and Levy, 1963; Simon, 1979; Shaikh, 1974). What these authors have shown is that the seemingly robust estimation results are due to the ‘Laws of Algebra’, i.e., practically any data can be fitted with a Cobb-Douglas like production function. Hence, Cobb-Douglas and CES production functions cannot be taken as valid representations of production in an economy.

These findings have been ignored by most of the profession. One reason could be that if a Cobb-Douglas can fit almost anything, it can fit production data that could be Cobb-Douglas. This tautology doesn’t need to be verified.

We have shown, see Fig. 3.3, that to convexify is not only improper methodologically, but it also leads to wrong conclusions with respect to efficiency and the estimates of potential output. A wealth of information on efficient heterogeneous methods of production is lost by resorting to convexification of the data.

In this article, we have computed the methods belonging to the wage-profit frontier and we have computed all the possible surrogate as-if aggregated values following the rigorous and robust methods as suggested in Samuelson (1962) (see eqs.2.20–2.21 and eqs. 2.22–2.24).

In the previous section, 3.3, we have verified whether the surrogate values have neoclassical properties. It has been shown that the as-if aggregate production functions are never neoclassical for the whole domain: the computed isoquants are never neoclassical and the capital-labor ratio is negatively sloped for the whole domain only for about 5% of the numéraires\(^\text{16}\). Furthermore, the elasticity of substitution, eq.3.4, is not constant and is very far from being equal to 1.

5 Conclusion

In this paper we have been able to verify whether the empirically generated data justify the artificial construction and use of the as-if neoclassical production function. The set of methods \(\Phi\), from which the most efficient methods \(Z^{WPF}_{E}\) have been extracted is not neoclassical in the sense that the aggregate values are far from being coherent with respect to set of assumptions described in section 1.

\(^{16}\)In Zambelli (2004) an investigation was conducted with virtual production systems, where it was shown that the capital-labor ratio would be negatively sloped 40% of the time. Here, with real data, it is shown that it is negatively sloped for only 5% of the time and it is never independent of the chosen numéraire.
The most important result is that it is not the case that at isoquants the *aggregate capital* is negatively related with the interest rate. This is the same as stating that the marginal productivity of aggregate capital is NOT always negatively related with the profit rate. That is, labor and capital demand functions are not *well-behaved*.

If the results presented here hold, this means that the standard notion of ‘fundamentals’ needs to be subjected to revision.

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Appendix A discussion on the limitations and potential criticisms

1. **High degree of methods of production & substitutability.** It could argued that the set of methods \( \Phi \) is composed of all possible methods or blueprints coming from nations that have different characteristics. To assume that these different methods could be combined so as to form the methods at the frontier \( Z_{\Phi E_{year}}^{\text{WPF}} \), where \( E_{year} \subset \Phi \) is a very strong assumption that requires a high degree of substitutability across national systems of production.

**Response.** The set of methods \( Z_{\Phi E_{year}}^{\text{WPF}} \) represent a benchmark. To view this benchmark as non-feasible because some methods of production only have local applicability is reasonable. Alternatively, one could pick a subset \( E'_{year} \subset E_{year} \subset \Phi \) and derive the set of methods at the frontier associated with it, \( Z_{E'_{year}}^{\text{WPF}} \). We have tried to construct subsets of \( \Phi \) with respect to regional proximity (for example North America, South America, East Asia, South Asia, North Europa, South Europa and so on) and the results are qualitatively similar to the earlier results. We have maximum degree of substitutability at the one extreme, as we have in this paper, and no substitutability on the other end. The latter would be the case in which the methods are nation-dependent, in which case we only operate with wage-profit curves. Also in this very simple case, we can question whether the system has properties that are necessary for it to be considered neoclassical. For reasons of space we do not deal with that issue here, but we this problem has been addressed in (Zambelli et al., 2014). Using data for 15 years and 30 countries, we have studied 450 different systems and their characteristics with variants of the aggregate values as with the eqs. 2.20–2.21 and eqs. 2.22–2.24, where we replace the wage-profit frontiers with the wage-profit curves. Even for the assumption of non-substitutability in the methods of production, the conclusion remains that there is empirical evidence in support of non-neoclassical properties.

2. **Fixed versus flexible proportions of the means of production.** Here we have considered the entries of the input-output tables, eq. 2.1, as methods of production (blue prints). There are two alternative ways to deal with the observations. One would be to consider each observation as separate or distinct liner fixed proportions method of production. The second would be to consider all the observations as points of the same production function, \( f \),

\[
 f(a_{i1}^{z_i}, a_{i2}^{z_i}, \ldots, a_{in}^{z_i}, \ell_i^{z_i}) = b_i^{z_i}
\]

where \( z_i = 1, ..., s_i \). In this case there would be \( s_i \) observations from which to estimate the above function.

**Response.** The important issue at hand is whether the production function \( f \) has the required neoclassical properties and not whether there is substitutability among means of production. The estimation of \( f \) has to be done appropriately and rigorously. To assume \textit{a priori} that \( f \) is a Cobb-Douglas or CES like production function is statistically improper (see Simon and Levy (1963); Simon (1979); Shaikh (1974)).

In our approach, we do not place any restrictions on the structure of \( \Phi \). Therefore, it is possible that the set of most efficient methods of production \( Z_{E'_{year}}^{\text{WPF}} \) could embed
substitutions in the means of production coherent with neoclassical assumptions. However, this is not the case. The isoquants are all of the type shown in fig. 3.1, and not like the ones in fig. 2.1. This shows that if the function \( f \) exists, it may not necessarily be neoclassical.

3. **Fixed versus Circulating Capital.** Neoclassical production functions have fixed capital and labor as inputs, while here we work exclusively with circulating capital, i.e. the means of production used during the year.

**Response.** Our measurement of capital is based on eq. 2.10

\[
K_{val}(r, x, \eta) = x^z A^z p^z(r, \eta).
\]

This might not be entirely satisfactory, but it is equally difficult to figure out alternative ways to measure capital. In the case of the aggregate neoclassical production function, capital is homogeneous with output. Estimates of fixed capital are usually based on the perpetual inventory methods, whose they are early flow values of the input-output tables. Although attempts to find robust indexes (or proxies) to measure capital (as if it was a physical magnitude) are made, the point of departure is at the transformations that directly use the input-output tables observations. A fraction of the value of output produced is assumed to be used for the fixed capital formation (investment), while a fraction \( \delta \) is the assumed to be the depreciation of capital. The question is whether the use of the factors of production should be negatively related as the factor prices change. This should occur for the circulating capital as well, which is a highly correlated component with respect to the total value of capital.

4. **Production prices are not market prices.** It might be argued that the prices that we use to compute the values of aggregate output and capital are not market prices, instead virtually generated prices and hence they are not relevant.

**Response.** Production prices are those that would be necessary for the system to be able to reproduce itself and they are the accounting equilibrium prices. Hence market prices should have values around these prices. Furthermore, these production prices should follow the scarcity principle as well. This means that with an increase in the profit rate, the relative prices of the more capital intensive sectors should decrease faster than the prices of the more labor intensive sectors. They should anyway change monotonically as the profit and the wage rates change.
References


