Algorithmic Social Sciences Research Unit

ASSRU
Department of Economics
University of Trento
Via Inama 5
381 22 Trento, Italy

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THE *CHIMERA*♠ OF A COMPLETE ANALYSIS OF ECONOMIC DYNAMICS♣

K. Vela Velupillai

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♠ 'A mere wild fancy; an unfounded conception. (The ordinary modern use), 1587. Also attrib.', according to **The Shorter Oxford English Dictionary: On Historical Principles**, italics added (the first of two volumes of which is only 1280 pages in length!).

♣The subject matter of this essay is squarely within the research program of the Algorithmic Social Sciences Research Unit (ASSRU), at the department of economics, University of Trento. I have benefitted greatly from discussions with my resident fellow ASSRU members, Professor Stefano Zambelli, Ms. Selda Kao and Mr V. Ragupathy. Alas, none of them agree to bear responsibility for the remaining infelicities, which must, therefore, be borne entirely by me.

Abstract

Alberto Quadrio Curzio's lifelong research efforts have been characterised by depth of vision, breadth of scholarship, a growing awareness of the need for sensitive mediation between theoretical analysis and empirical, policy oriented, concerns and, above all,the need for underpinning every kind of aim with an institutional and historical perspective. This contribution to a **Festschrift** in his honour aims to take up one small aspect of one of his many faceted theoretical concerns. Alberto Quadrio Curzio's conviction that there can be *no 'complete analysis' of economic dynamics – not even a 'complete' formalisation* of this tortuous notion – is given some content (albeit a very minor one) in this paper.

1 A Personal Preamble

> "[Non-produced means of production (NPMP)] influence prices and distribution, the relations between these two entities and production, the *continuous*[1] but *non-regular growth and dynamics of economic systems* and international trade. This [conviction that reality lies between the pessimism of Ricardo and the optimism of an eminent father of contemporary dynamic analysis] tells us that [NPMPs], in their coexistence with produced and reproduced means of production, influence the distribution of income and have effects that range from *a non-regular dynamic of economic systems*, which *continue* to grow, to special international economic relations. By following this path of *constructive research* it will also be possible to utilize part of the theories of production of Leontief, with its *formidable empirical content*, and of von Neumann."
>
> Quadrio Curzio (1980), pp.219,239; italics added.

My first encounter with one of Quadrio Curzio's brilliant writings was one that was bristling with implicit dynamics, although clothed in the impeccable and illuminating formalism of capital theory – and that was over three decades ago (Quadrio Curzio, 1980). Over the next two decades – and more – Quadrio has worked out the 'formidable empirical content' of the clearly set out theoretical research program (Quadrio Curzio & Pellizzari, 1999, 1996), 'following [a] path of constructive research', implicit in that classic, itself a continuation of a deeply original line of research, powerfully inspired in an unparalleled way by Sraffa's classic (Sraffa, 1960), that had begun at least as early as 1967 (Quadrio Curzio, 1967, 1975).

In a precisely definable sense of the consummation of a research program, 'following a path of constructive research' and endowing it with some 'formidable empirical content', the more mature Quadrio concluded (AQC & FP, 1999, p.240; italics added), almost with some resignation:

> "In conclusion . . . . we recall some studies of economic dynamics just to remind us [of] the vast range and complexity of this field, and the *absence* of a *complete analysis of economic dynamics*."

Perhaps one should read 'absence' as '*impossibility*'! Such a reading would be entirely consistent with the view of Siro Lombardini, summarised, I think with warm approval by AQC & FP *(ibid*, p. 242; italics added):

> "Lombardini ... reminds us that 'the' model of economic growth *does not exist*. ... Lombardini also reminds us that the mathematical *complexity* of these models always risk to transform theory in a pure formal exercise. .. In conclusion ... Lombardini asks if there exists

[1] The unwary reader may be tempted to interpret 'continuous' in its ordinary (real analytic) mathematical sense, but the later phrase 'continue to grow' should dispel such a person of any such illusion.

> 'the true' theory of growth. His answer is *negative. Theory always* takes a specific definition of [an] economic system and *chooses a select set of tools and methods for its analysis.* But all definitions are partial and *each tool or method can clarify some specific aspects of an economic system, leaving aside others.*"

These wise caveats by Quadrio Curzio are those of a scholar with a mastery of the history of thought, a supreme economist who has grappled with the conundrums of grounding empirical content in solid economic theory, a concerned policy analyst who sees the need for understanding the evolution of institutions, and a passionate advocate of the need to foster innovation and, therefore, to grasp the role of entrepreneurship in advanced industrial economies (Quadrio Curzio & Fortis, 2005, 2008). In these attempts at forging an eclectic vision for an enlightened society, Quadrio Curzio invokes, as if they were intimate friends with whom he converses regularly, the classics and the pioneering Italian pre-Classics (for example, Cesare Beccaria and Ferdinando Galiani), our neo classical masters, and those of the interregnum who aimed to consolidate the classical framework - John Stuart Mill and John E. Cairnes - and some who went beyond, Karl Marx, and the giants of the 20th Century, from Wicksell and Schumpeter, to Keynes and Sraffa. There were the Scottish and the French enlightenments - and also the partial Italian ones, those of an old nation yet to be forged into a new country. These partial Italian contributions, particularly the *Lombard enlightenment*, are reflected in Quadrio Curzio's passionate commitment to the unified ideals of European values, against a backdrop of European unity and unified policy for research and innovation within a framework of evolutionary institutions, fostering entrepreneurship (cf., Quadrio Curzio & Fortis, 2005, 2008)[2].

I have often found myself remembering, and reminding my own students, when we are prone to indulge in flights of theoretical fancy and atheoretical empirical fantasies, precepts elegantly and concisely formulated by Quadrio Curzio, but in the unlikely context of 'the Gold Problem' (Quadrio Curzio, 1982, p.1):

> "History and fantasy, scientific analysis and platitudes, present-day dicta and the beliefs of centuries past overlap and mingle in such a way as to render quite complex both an objective analysis and a detached reading of contributions which seek to be objective."

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[2] In my review of the book by Lombardini (1996), which is summarised as above by Quadrio Curzio, I wrote (Velupillai, 1997, pp. 1260-1):

> "In sum, these chapters are an elegant, but rigorous, plea for an amalgam of the visions advocated by Schumpeter and Simon on the foundations built by our classical and neoclassical predecessors. It is not about micro and macro with the former providing the underpinning for the latter. It is, rather, about entrepreneurs and innovation, about evolution and institutions, about adapting and computing, about procedures and policies, and about applying Marshall's epitaph in endogenising these features on the structures that we have inherited from Smith and Ricardo, from Walras and Pareto."

I would feel entirely comfortable in characterising Quadro Curzio's visions in the above vein, to which my only addition would be the name of *Piero Sraffa*.

Alberto Quadrio Curzio, an *aristocrat* of theory, with a mastery of history and tradition – as Carlo Cipolla was, in a not unrelated field – combined majestically, as I, with my stunted scholarship and blinkered visions have understood, the visions of a Schumpeter with the theoretical purity of a Sraffa, and took that synthesis to new heights of formidable empirical and institutional scholarship. Lesser mortals like me can only wonder, with admiration and seek to emulate, the ideals of scholarship, institution-building and fostering of the evolution of tradition that Alberto Quadrio Curzio has made into an art, whilst also personifying its encapsulation in one, whole, personality.

2 *Complete* Analysis of Economic Dynamics

> "The final aim of the theory of the motions of a dynamical system must be directed toward the qualitative determination of *all possible types of motions* and of the interrelation of these motions.
>
> Birkhoff, 1927, p. 189; italics added

The obsession with the desire for a *complete analysis of economic dynamics*, is analogous to the classic - but forlorn - search for a 'qualitative determination of *all possible types of motions*', or to considering all possible sets and ending up with the (in)famous Russell paradox of considering all possible sets and wondering what to do about that class which is made up those sets that are not members of themselves.

It is possible, perhaps even desirable, to separate the notion of *complete analysis* of any problem – not only in economics – and the idea of a complete economic dynamics, independent (at least in the latter case) of the feasibility of its complete analysis. Then Quadrio Curzio's wise injunction comes into its own (see above and AQC & FP, 1999, p. 240):

> '*Theory always* takes a specific definition of [an] economic system and *chooses a select set of tools and methods for its analysis.*'

That 'set of tools and methods for its analysis', chosen by a 'theory taking a specific definition of [an] economic system' is often thought of as having achieved completeness – of analysis, of formalisation, etc. Thus in the admirable volume of essays in honour of Augusto Graziani (Arena & Salvadori, 2003)[3], Wynne Godley claims (*ibid*, p. 143) to have built a Complete Keynesian Model (albeit a simple one!). Even more remarkably, in the same volume, Jean Cartelier claims that (ibid, p. 221; italics added):

> "A *complete* demonstration of the existence of a Keynesian equilibrium in an Arrow-Debreu framework exists."[4]

[3] In which there is also a fine, concise, summary of AQC & FP, 1999 (Quadrio Curzio & Pellizzari, 2003).

[4] He then goes on to refer to the 1968 work of Glustoff, adding: 'It seems that no Keynesian has ever been aware of it.'! I am not sure who he means by 'Keynesians', but it was fairly

Obviously Professor Cartelier, like most of the profession, has forgotten – or, more likely, ignorant of the fact – that Glustoff (like his fellow general equilibrium theorists) 'chose a select set of tool and methods' to provide a so-called '*complete* demonstration of the existence of a Keynesian equilibrium in an Arrow-Debreu framework'. From the perspective of a different kind of mathematics, say computable or constructive analysis, this demonstration by Glustoff would not be considered complete[5]. This is precisely the point made by Quadrio Curzio - leaving aside the other injunction of taking 'a specific definition of [an] economic system', often unadaptible to the tools and methods of analysis being used.

To the best of my knowledge, the earliest attempt to seek a formalisation of a notion of *complete economic dynamics* was in a classic paper by Kuznets at the dawn of the era which sought to integrate equilibrium theory with dynamics (Kuznets, 1930). On the other hand, all attempts, in every kind of economic theory, at formulating 'General Theories' – be it the *General Theory of Employment, Interest and Money* or *General Equilibrium Theory* – are efforts, always falling foul of Quadrio Curzio's above wise injunctions, at building *complete* theories and choosing a select set of tools and methods for its analysis, from which special theories are derived.

Consider the following *general* form of a differential-difference equation of differential order n and difference order m (Bellman & Cooke, 1963, p. 43):

$$F\left[t, u(t), u(t-\omega_1)\,..,u(t-\omega_m), u'(t), u'(t-\omega_1), .., u^{(n)}(t), u^{(n)}(t-\omega_n)\right] = 0 \quad (1)$$

Where:

F and u are real functions of real variables, t; ω_i: real numbers, $\forall i = 1, ...m$; and n, m : integers.

Then it is easy to show that the endogenous macrodynamic business cycle models of Kalecki (1935), Lundberg (1937), Kaldor (1940), Hicks (1950) and Goodwin (1951) can be derived from the above general form of a differential-

well known among the cognoscenti working on varieties of fix price equilibria from the early 1970s all the way till that whole approach died the natural death it deserved to die of, by the early 1980s.

[5] The obverse side of this absurd Cartelier coin is the claim by Kaldor (1954, p. 53; italics added):

> "*[I]t is not possible* to make the [Schumpeter] story as a whole into a 'model' (meaning by a model the sum total of assumptions which are just sufficient – no more no less - together to provide the necessary and sufficient conditions for the generation of a recurrent cycle with a clear periodicity) without incorporating into it elements which would suffice by themselves to explain the cycle – without recourse to Schumpeter's own stage army of initiators and imitators, or even the very concept of techincal progress."

How does Kaldor know that 'it is *not possible* to make the [Schumpeter] story as a whole into a model'? Is this an 'impossibility theorem', *within some mathematical formalism* of theories and models?

difference equation. Now two simple questions can be posed:

1. In what sense can (1) be considered a *complete* formulation of endogenous business cycle theories? In other words, do differential, difference and mixed differential-difference equations encapsulate, completely, *all dynamic phenomena in economics*?

2. Even if we grant the above, is there *one unique mode of analysis* of such a system for which *completeness* can formally be defined?

In answer, first, to the second part of the first question, it is easy to construct an example of an algorithmic dynamical system that has not been shown to be reducible to the dynamics of a differential, difference or mixed differential-difference system of any generality. I have in mind here the Goodstein Process (cf. Goodstein, 1944 and Paris & Tavakol, 1993). As for an answer to the first part, nothing in dynamical systems theory indicates that there is any hope of characterising the complete (sic!) dynamics of (1), when hardly any breach has been made towards even solving the second part of *Hilbert's 16th Problem* regarding even planar dynamical systems (Hilbert, 1902).

As for the second question, first of all there are many modes of analysis to *prove* the existence of a closed path using the *Poincaré-Benidxson theorem*, as applied, for example, to Kaldor (1940), some of which invoke Hilbert's notion of 'consistency implies existence' which fall foul of Gödel's results on *incompleteness* (Shapiro, 1997, p. 134)[6]. Moreover, it is by now well known that many special cases of (1) are subject to *undecidable dynamics* (Stewart, 1991), which implies that *no* constructive, algorithmic, *methods* can lead to any kind of complete analysis, except in the world of non-applicable (i.e., non-empirical) classical mathematics.

3 *If* Economic Dynamics *is* Algorithmic Dynamics, *then Complete* Dynamics is a Chimera

But all the clocks in the city

Began to whirr and chime:

'O let not Time deceive you,

You cannot conquer Time.

......

Time watches from the shadow

And coughs when you would kiss.

[6] *A fortiori*, all orthodox mathematical general equilibrium theories are subject to the (second) *incompleteness theorem of Gödel* (Velupillai, 2009). I am afraid this also applies to formal Neo Ricardian Models of growth and distribution – although it is **not** applicable to Sraffa's own framework of analysis in his *magnum opus* (Sraffa, 1960).

'In headaches and in worry

Vaguely life leaks away,

And Time will have his fancy

To-morrow or to-day.

W.H. Auden: As I walked Out One Evening[7]

Change, in economics and in all the sciences, is considered synonymous with dynamics, at least since Newton and always in mathematical formalisms that invoke one or another form of dynamical systems theory. Most commonly, almost trivially, in terms of differential, difference or mixed differential-difference systems of equations. These phantoms lead to the false dichotomy, in economics, mimicking that which originated in Newtonian mechanics, between statics and dynamics; between equilibrium and disequilibrium; and between stability and instability. Rarely does one think - at least not in economics – of *algorithms* as dynamic objects, implemented in time, imperative in their logical form and intrinsically constructive, in the strict mathematical sense of this word, especially in its Intuitionistic, Brouwerian, versions. As affirmed in what he called the *First Act of Intuitionism*, Brouwer stated, in his **Cambridge Lectures on Intuitionism** (Brouwer, 1981, p. 4; italics in the original) uncompromisingly:

> "Completely separating mathematics from mathematical language and hence from the phenomena of language described by theoretical logic, recognizing that intuitionistic mathematics is an essentially languageless activity of the mind having its origins in the perception of a move of time. This perception of a move of time may be described as the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory."

The fundamental dichotomies in algorithmic mathematics – whether it be in its computable or constructive versions – is that between the halting and non-halting behaviour of a machine implementing an algorithm, between decidable and undecidable problems that an algorithm aims to solve, against the backdrop

[7] St. Augustine's famous reflections on TIME, in Bk. XI of his Confessions, is worth remembering as we, in economics, think we can master it by one or another kind of formalisation:

> "What, then, is time? If no one asks me, I know; if I wish to explain to him who asks, I do not know. Yet I say with confidence, that I know that if nothing passed away, there would be no past time; and if nothing were coming, there would be no future time; and if nothing were, there would be no present time. Those two times, past and future, how are they, when even the past not now, and the future is not yet? But should the present always be present, and should it not pass into time past, it could not truly be time, but eternity. If then, time present–if it be time–only comes into existence because it passes into time past, how do we say that even this is, whose cause of being is that it shall not be, namely, that we cannot say what time truly is, unless because it tends not to be."

of the further dichotomy between the solvability or the unsolvability, easily or with difficulty, of a decision problem[8].

Six decades ago, at the pre-dawn period of what eventually became the algorithmic avalanche and the dominance of the computational[9] approach to economics, in both theoretical and empirical modes, Richard Goodwin's acute prescience led him to observe that:

> "In the continuing effort to erect a useful dynamics [in economics], we need all possible helpful sources or hypotheses. One such is the traditional mathematical device of solving equations by trial and error and its vigorous modern step-child the automatic computing machine or zeroing servo.
>
> It always takes time to solve an equation by approximative methods, and if our methods for determining the successive approximations are made analogous to the structure of economic decisions, then we may regard the sequence of steps as entirely parallel to an actual process of economic dynamics in time. The convergence of the approximations to the correct answer is the same as the dynamical stability of the economy. ... Hence a solution corresponds to an equilibrium value. Similarly we may regard economic dynamics as such a series of iterated trial solutions ... .
>
> [C]ontinuous [economic] processes (with differential equations) may .. be regarded as trial and error *methods of solution* to static equations. The reason why it is not so easy to see is that no human being can make continuous trials infinite in number. This gap in our grasp of the problem has been closed by the perfection of electronic and electro-mechanical computers .. . Such a machine is an exact analogue of a continuous dynamic process[10]. Therefore it seems entirely permissible to *regard the motion of an economy as a process of computing the answers to the problems posed to it.*"
>
> Goodwin, 1951a, pp. 1-2; italics added.

In this perceptive conjecture, Goodwin suggests an equivalence between *algorithmic dynamics* and *economic dynamics*, mediated by viewing 'the motion of an economy as a process of computing the answers to the problems posed to it'. The false dichotomies pointed out above are replaced by those that are intrinsic to algorithmic dynamics.

[8] Again, interpreted in its strict metamathematical sense. It may be useful to point out that standard optimization problems, as in economics and operations research are trivial special cases of decision problems.

[9] Not necessarily to be equated with the notion of *computability* in its recursion theoretic senses.

[10] Goodwin is, here, referring to analogue computers. However, such computers do not have more power, in the sense of computing beyond the *Turing Limit,* by 'breaking' the strictures of the *Church-Turing Thesis.* Hence, I shall assume we are working with ideal digital computing machines – i.e., *Turing Machines* – in the discussions of this section. These results are discussed, described and explained in Velupillai (2011).

Computation, *problem*, *solvability* and *dynamics* are all defined algorithmically.

However, algorithmic dynamics is replete with undecidabilities and incompleteness, in the strict metamathematical senses; it is also richly endowed with uncomputabilities in recursion theoretic senses. Moreover, carrying the suggested equivalences a little further, it would be natural to ask what kind of *problems* can be posed to 'such a machine' (or the economic system) and ask whether we can classify the 'posable' problems as *easy* and *hard*, on a sliding scale, first separating the *decidable* problems from the *undecidable* ones. Such questions lead immediately to structuring the data, which is the bridge between theory and empirical implementation, in terms of sets of numbers that are recursive, recursively enumerable and recursively enumerable but not recursive. These, then, lead to notions of recursive separability and inseparability, from which *naturally* paradoxical undecidable and uncomputable dynamics emerge, encapsulable even in the formalism of dynamical systems theory (Smullyan, 1993; Pour-El & Richards, 1989).

The classification of the *decidable problems* – i.e., classification of recursive sets – into a sliding scale, ranging from easy to hard - the domain of *computational complexity theory*[11] – say from *polynomial time computability* at the 'easy' end, all the way to the NP-*Complete* problems at the difficult end.

On the other hand, classification of undecidable problems – i.e., classification of non-recursive sets (of numbers) – requires the formalism of *oracle* (or *relative*) computations, *reducibilities* and a rigorous notion of *complete* (Davis, Sigal & Weyuker, 1994; Salomaa, 1985).

The idea of classifying *undecidable problems* in terms of the difficulty of *solving them* may seem like a contradiction in terms. However, this apparent 'contradiction' is resolved by means of Oracle Turing Machines, or using relative computations (Rogers, 1967; Davis, 1958). Thus, an undecidable problem P^* is classified as being *at least as difficult* as another problem, P, if an algorithm, to which an Oracle is adjoined, implemented on an Oracle Turing Machine, that can solve P^*can also be utilized to solve P. If the reverse process is infeasible, i.e., if the undecidable problem P, solved with an Oracle Turing Machine, cannot be used to solve P^*, then the latter is, intuitively, considered 'more undecidable', or more difficult, than the former.

Definition 1 *Given two sets of arbitrary – i.e., not necessarily recursively enumerable sets – nonnegative integers, S and S^*, if any algorithm for solving whether an element is a member of S^* implies the existence of an algorithm for solving the membership problem for S, then we denote this by:*

$$S \leq S^* \tag{2}$$

$$If\ S \leq S^*\ \&\ S^* \leq S\ then\ S \equiv S^* \tag{3}$$

[11] Perhaps the interested reader could be referred to my attempt at providing a pathway into computational complexity theory (Velupillai, 2008) for the economist.

Definition 2 *Complete: If S^* is a recursively enumerable set, and $\forall S$, also recursively enumerable sets, $S \leq S^*$, then S^* is said to be* ***COMPLETE****.*

Definition 3 *m-reducible: S is $m-reducible$ to S^* iff $\exists$ a recursive function g s.t., $\forall$ i, $i \in g(i)$ in case $g \in S^*$*

Remark 4 *Every recursively enumerable set is $m-reducible$ to the $Halting$ set.*

Thus the *halting problem for Turing Machines* is an alternative mode of tackling undecidability in the context of recursively enumerable sets and their membership problem. If the economic system's dynamics is to be interpreted as the trajectory of the path taken by a computing machine solving problems posed to it, then the data sets – in this case, economic data in the form of numbers – must be considered to be appropriately recursive. *Ad hoc* habits of assuming data to be generated from exogenous probability distributions, or worse, will have to be given up. In this way theory and empirical analysis is tied together within a unified algorithmic framework.

This notion of *complete* has, of course, nothing whatsoever to do with the loose, almost flippant, use of the word *complete* by economists, claiming to model this or that completely, or analyse a given model completely; not even anything to do with the notion of completeness in classical real analysis (for example the 'completeness of a metric space'); above all, nothing at all to do with the deep notions of completeness and incompleteness in metamathematics, due to the pioneering contributions of Thoralf Skolem and Kurt Gödel in the first third of the 20th century.

If Kaldor can imagine an *Economics without Equilibrium* (Kaldor, 1985), I see no reason why, with a vision from an algorithmic point of view, I should not – or cannot – imagine an economics without the dichotomies that originated in the natural sciences and replace them with those that are intrinsic to a mathematics that is naturally adapted to implementing empirically the theoretical structures of economics.

Completeness, even in mathematics, has become a weaselword; disentangling its many splendoured variations, showing its chimerical usage by pseudo-mathematical economists, is itself a difficult task, like that faced by a mycologist trying to classify mushrooms, as Vladimir Arnold suggested:

> "When you are collecting mushrooms, you only see the mushroom itself. But if you are a mycologist, you know that the real mushroom is in the earth. .... In mathematics, the upper part of the mushroom corresponds to theorems that you see. But you don't see the things which are below, namely problems, conjectures, mistakes, ideas, and so on.
>
> You might have several apparently unrelated mushrooms and are unable to see what their connection is unless you know what is behind. And that's what I am now trying to describe. This is difficult, because to study the visible part of the mathematical

mushroom you use the left half of the brain (which deals with logic), while for the other part the left brain has no role at all, because that part is extremely illogical. It is therefore difficult to communicate it to others. But here I shall try to do it."

Arnold (2006), pp.19-20; bold italics, added.

Taking a cue from what I like to think of as Alberto Quadrio Curzio's fundamental precept – to recognise, explicitly, that '*theory always* takes a specific definition of [an] economic system and *chooses a select set of tools and methods for its analysis*' – I have tried to discuss what may be implied by selecting a particular set of tools and methods for the analysis of economic theory. I have tried to couple this to a vision of an economic system devoid of false dichotomies, those that have shackled economic theory for over two centuries. The price of the vision I advocate is the introduction of new dichotomies, but those that I think are natural to economic analysis in its mathematical mode.

However, anything I can contribute, to pay *homage* to the *aristocratic scholar-gentleman that is Alberto Quadrio Curzio* must, surely, be woefully inadequate from every point of view. Yet, I consider myself privileged to have been invited to do so, and - however inadequate - it is written with conviction and humility, knowing that the unattainable is also the impossible, but yet, with Tennyson's **Ulysses**:

'T is not too late to seek a newer world.
...
It may be that the gulfs will wash us down;
It may be we shall touch the Happy Isles,
And see the great Achilles, whom we knew.
Tho' much is taken, much abides; and tho'
We are not now that strength which in old days
Moved earth and heaven, that which we are, we are,–
One equal temper of heroic hearts,
Made weak by time and fate, but strong in will
To strive, to seek, to find, and not to yield.

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