Algorithmic Social Sciences Research Unit

ASSRU

Discussion Paper Series

7 – 2016/I

The Aggregate Production Function is NOT Neoclassical

Stefano Zambelli

November 2016

The Aggregate Production Function is NOT Neoclassical

Stefano Zambelli*

Abstract

Standard postulates concerning the aggregate production function are about marginal productivities and the associated demands for labour and capital. These demands are supposed to be negatively related to the factor prices, namely the *wage rate* and the *interest rate*. The theoretical cases in which these neoclassical properties fail to hold are regarded as anomalies. We compute the aggregate values for production, capital, and labour and find that the neoclassical postulates do not hold for the detailed dataset that we consider. The obvious implication of this result is that the models and analysis based on the aggregate neoclassical production functions are ill-founded as they are based on something that does not exist.

Keywords: Aggregate Neoclassical Production Function, Cobb-Douglas, CES, Technological Change, Macroeconomics, Growth.
JEL classifications: E1, 04, C3

*Department of Economics and Management, University of Trento; via Inama, 5, 38122 Trento, Italy. E-mail: stefano.zambelli@unitn.it

Introduction . Aggregation and the Neoclassical Postulate.

It is a widespread practice among economists to use the *'neoclassical'* aggregate production function, especially while constructing macroeconomic models[1]. These models often represent an economic system producing a large number of heterogeneous goods in terms of a few index numbers - one each for output Y, productive capital K, total quantity of employed labour L, and technological level or knowledge A[2].

Samuelson (1962, p.194 fn.1) has appropriately referred to this type of aggregate production functions as the *surrogate* or *as-if* production function. Such an aggregate representation may be useful, provided that the indexes have certain properties (Fisher, 1922; Frisch, 1936)). However, at the end of the 1960s, it was concluded that aggregation could be problematic since there are cases where the aggregate (and also the dis-aggregate) production function may not have the desirable neoclassical properties.

The problems are of two types. The first concerns the technical aggregation from micro-level to macro-level. That is, simple *'well-behaved'* production functions, after aggregation, do not retain the same functional forms as before aggregation (Fisher, 1969; Shaikh, 1974)[3].

The second is known as the value problem, which was addressed during the two Cambridges debate[4]. The conclusion of this debate was that there are ~~exist~~ cases in which an aggregation from a multi-commodity space to a single *surrogate production function* (see Samuelson (1962)) may result in a production function that is not well-behaved (also admitted by Samuelson (1966))[5]. Solow, who acknowledged the problem, observed:

> ...I have to insist again that anyone who reads my 1955 article [Solow (1955)] will see that I invoke the formal conditions for *rigorous aggregation* not in the hope that they would be applicable ...but rather to suggest the *hopelessness of any formal justification of an aggregate production function in capital and labour* (Solow, 1976, p.138).

Despite a widespread acknowledgement that aggregation could be problematic, the (generalized) Cobb-Douglas production function is widely used in various models across theories and remains a fundamental tool for the empirical assessment of technological progress and productivity growth.

There could be two possible reasons for this. First, even though it is not assured that the aggregation will always preserve neoclassical properties, there exist, at least in theory, sets of methods for which a *neoclassical surrogate production function* does in fact exist. A distinction needs to be made between something which is logically possible (i.e., the impossibility of *any formal justification of an aggregate production function in capital and labour*, as Solow himself would admit) and the actual occurrence of that logical possibility. It could just be a *curiosum*, *paradox* or a *perversity*. Clearly, deciding whether or not we live in a neoclassical world without actual empirical evidence would merely be a matter of belief. Second, it is possible that the economists who questioned the validity of the neoclassical aggregate production function have not been able to convincingly demonstrate the empirical untenability of the Cobb-Douglas type (or the generalized CES type) production functions. They have also not been able to, or atleast so it appears, provide a valid, useful alternative.

In this paper we attempt an empirical inquiry into the tenability of the *neoclassical postulates* that are implied by use of the *neoclassical* production function. Ferguson, a neoclassical economist, conveys this very clearly:

> *[Neoclassical Postulate] ...the lower the rate of interest, the greater the capital intensity of production. All other neoclassical properties follow immediately from this simple relation* (Ferguson, 1969, p.252)

Many neoclassical economists believe that this postulate holds. It is at the foundation of all neoclassical economic models and it is found in almost all leading textbooks, often in the first few chapters. In our view, Sato's views about this postulate that were expressed years ago are still pertinent and reflect the state of affairs prevalent among the majority of economists today. He argued:

> ...that there is a *not-too-small* world in which the neoclassical postulate is perfectly valid. So long as we live in that world, we need not to give up the neoclassical postulate. In order to refute it, *it is necessary to demonstrate that this world is imaginary. This demonstration has not been supplied in the literature*. Nonetheless, it is important to realize that *there is another world in which the neoclassical postulate may not fare well or is contradicted. An empirical question is which of the two models is more probable*(Sato, 1974, p.383, italics added). .

On the one hand, Sato admits to the existence of the problem. On the other hand, however, he declares a belief that an empirically non-negligible proportion of the world does have *neoclassical* properties. In doing so, he portrays the problem of aggregation as a mere *curiosum*, which may be interesting from the theoretical point of view, but irrelevant for empirical applications. This position has not been satisfactorily challenged empirically with a conclusive demonstration concerning whether or not the world is neoclassical. This *de facto* empirically unchallenged belief that the world is *neoclassical* has led to a state of affairs in which productivity measurements (total and multiple factor productivities), measurements of technological progress and economic efficiency, in general, are all based on the neoclassical production function. In fact, the explicit or implicit points of departure for these measures are still the works of Solow (1957), Farrell (1957).

The main objective of this paper is to provide an empirical verification regarding whether the intensity of capital per unit of labour decreases (or increases) as the profit (or wage) rate increases. We find that there are no theoretical or empirical grounds to assume that the neoclassical production function would hold.

In Section 1 we discuss Samuelson's *surrogate* or *as-if* production function. It shows the possibility to go from a multiple-product, multiple-industry production system to a virtual *as-if* or *surrogate* aggregate production function. In subsection 1.1 we provide an operational definition of the technological set, the *Book of Blueprints*. In subsection 1.2 we show how to link *heterogeneous production*, relative to a particular choice of *blueprints* with *wages*, *profits* [6] and *production prices*. The fundamental notions of *wage-profit curves* and that of the common measurement unit, the *numéraire*, are introduced. In the Appendix A we present the standard neoclassical notion of the aggregate production function. It is meant to be a summary of the neoclassical production functions provided by several neoclassical scholars (Solow, 1955, 1956, 1957; Arrow et al., 1961; Ferguson, 1969; Shephard, 1970).

In subsection 1.3 we introduce the notion of *wage-profit frontier*, which is a combination or composition of a small subset of (efficient) *wage-profit curves* selected from the huge number of *wage-profit curves* associated with the astronomical number of possible combinations of different production methods. Associated with the *wage-profit frontier*, we have a small subset of blueprints, production methods. In subsection 1.4, we provide a definition of heterogeneous physical surplus, heterogeneous physical capital and then show how to use production prices to compute *aggregate* variables that are isomorphic to the values that we find in *national accounting*. Finally, we present an operational notion of *aggregate neoclassical production function* in the subsection 1.5. Following Samuelson (1962), we show the properties that the aggregate or homogeneous *as-if* or *surrogate* production function should have.

In section 2, we discuss the methodology to empirically validate (or negate) the *neoclassical postulates*. A discussion of the existing literature and empirical work is provided. We critically examine the previous empirical work on this subject and we argue that focusing exclusively on the shape the wage-profit curves, instead of the actual determination of the value of capital (using production prices directly), is quite incomplete and that leads to inconclusive evidence.

In Section 3 we describe the data we utilize and the rationale behind some of the choices (for example, the choices of the *numéraires*) that are made in this paper . Section 4 contains the results. We have used four different criteria to arrive at our conclusion. First, subsection 4.1, we check whether there is substitution of physical capital with labour, assuming a fixed physical surplus vector. In this case we preserve the intrinsic heterogeneous nature of production. Second, subsection 4.2, we compute the value of capital intensity per labour with the change in profit rates by keeping the physical surplus constant. Third, subsection 4.3, we compute the value of capital intensity per labour by keeping the value of the net national product fixed (and by minimizing the aggregate value of capital used). Fourth, subsection 4.4, we compute the value of capital indirectly, that is without using production prices, through the knowledge of the curvature of the *wage-profit frontier*. Note that all the criteria above are directly connected to the idea that there is a substitution of capital and labour as the profit rate increases. In concluding, section 5, we discuss the main result of the paper, which is, paraphrasing Sato (1974, p.383) : the *world is not neoclassical*.

1 Surrogate Aggregates: Output, Capital and the Production Function

1.1 Technological set as *"Book of Blueprints"*

An important question concerning the economic system is whether it is sound to assume that a production system that consists of many products and different methods of production can be represented using a simple, well-behaved, aggregate neoclassical production function. Samuelson (1962) proposed a method that is meant to provide a theoretical justification for the simplification, starting from a dis-aggregated, industry-level production systems.

> One need never speak of *the* production function, but rather should speak of a great number of separate production functions, which correspond to each activity and which need have no smooth substitutability properties. All the technology of the economy could be summarized in a whole book of such production functions, each page giving the blueprint for a particular activity. Technological change can be handled easily by adding new options and blue prints to the book ... Finally it is enough to assume that there is but one 'primary' or non-producible factor of production, which we might as well call labour ... All other inputs and outputs are producible by the technologies specified in the blueprints (Samuelson, 1962, p.194)

Samuelson proposed a way to link a system close to reality, where there is heterogeneous production, with a fictitious or *"surrogate"* or *"as-if"* or *"fairy tale"* equivalent.

> Now let us forget our realistic book of blueprints. Instead suppose labour and a homogeneous *capital jelly* (physical not dollar jelly!) produce a flow of homogeneous net national product, which can consist of consumption goods or of net capital (i.e. jelly) formation, the two being infinitely substitutable (in the long run, or possibly even in the short run) on a one-for-one basis. The resulting production function obeys constant returns to scale and may have smooth substitutability and well-behaved marginal productivity partial derivatives. Such a Ramsey model, if it held, could justify all of Solow's statistical manipulations with full rigour.
>
> As is well known, labour's share is given by total labour times its marginal productivity. The marginal productivity of capital (jelly) tells us how much a unit of the stock of capital can add to its own rate of capital formation per unit time: the result is the (own) rate of profit or interest, a pure number per unit time ...
>
> ... Indeed if we invent the right *fairy tale* , we can come as close as we like to *duplicating the true blue-print reality in all its complexity*. The approximating neoclassical production function is my new concept of the *Surrogate Production Function* ((Samuelson, 1962, pp.200-201, emphasis added).

The *book of blueprints* can be seen as different entries of the input-output tables[7]. We observe from the actual tables that b_i units of commodity i can be produced with s_i different alternative combinations between inputs and outputs.

$$\phi_i(z_i) : a_{i1}^{z_i}, a_{i2}^{z_i}, \dots, a_{in}^{z_i}, \ell_i^{z_i} \mapsto b_i^{z_i} \tag{1.1}$$

where: $i = 1, \dots, n; j = 1, \dots, n; \quad z_i = 1, ..., s_i$. $a_{i1}^{z_i}$ is the input of commodity j necessary for the production of good i using the combination (method or observations) z_i as inputs. s_i is the number of available observed combinations for producing good i and n is the number of goods[8].

The set of observed combinations for producing good i – i.e., the set of blueprints for the production of i – is composed of all alternative production methods $\Phi_i = \{\phi_i(1) \cup \phi_i(2) \cup \dots \cup \phi_i(z_i) \cup \dots \cup \phi(s_i\}$. The set of all available methods, the book of blueprints, is given by the following set of activities $\mathbf{\Phi} = \{\Phi_1 \cup \Phi_2 \cup \dots \cup \Phi_n\}$. Hence, a n-commodity output vector can be generated by using each combination of methods, which belongs to set $\mathbf{\Phi}$. There are a total $\boldsymbol{s} = \prod_{i=1}^{n} s_i$ of these combinations. This, even for a small set of alternatives, is an astronomical number (see footnote 13). Given one of these combinations, $\mathbf{z} = [z_1, z_2, \dots, z_n]'$, we have one production possibility. The heterogeneous production of a system would then depend on the level of employment and the methods of production adopted. The quadruple ($\mathbf{A^z}$, $\boldsymbol{L^z}$, $\boldsymbol{B^z}$, $\mathbf{x}$) is the standard representation of an input-output system, where $\mathbf{A^z}$ is the set of the inputs used, $\boldsymbol{L^z}$ is the vector of the amount of labour that is necessary and $\boldsymbol{B^z}$ is the associated output [9] and $\mathbf{x}$ is the vector defining the level of activity[10].

1.2 The wage-profit curves, the production prices and the *non-substitution theorem.*

For a chosen system, $\mathbf{z}$, (i.e., a triple $\mathbf{B^z}, \mathbf{A^z}, \mathbf{L^z}$, $\mathbf{x}$) the *production prices* that would assure the accounting equilibrium are those that allow the following relation to hold[11]:

$$\mathbf{A^z}(1+r)\mathbf{p} + \mathbf{L^z}w = \mathbf{B^z}\mathbf{p} \tag{1.2}$$

For a given uniform profit rate r and a uniform wage rate w, there exists a price vector $\mathbf{p}$ that would allow the system to remain productive for the subsequent periods as well :

$$\mathbf{p^z}(r, w) = [\mathbf{B^z} - \mathbf{A^z}(1+r)]^{-1}\mathbf{L^z}w \tag{1.3}$$

Here it is important to recall that the production prices, wage rate and the profit rate are not dependent on the level of the activity. They are not dependent on the level and composition of the generated surplus. This seemingly counter-intuitive result is known as the *non-substitution theorem*[12] . On the origins of the *non-substitution-theorem*, see Arrow (1951); Koopmans (1951); Samuelson (1951). A more recent treatment is given in Mas-Colell et al. (1995, pp.159-60).

We then choose a *numéraire* η, a vector composed of different proportions of the n

produced goods forming the input-output tables,

$$\eta' \mathbf{p}^{\mathbf{z}}(r, w) = 1 \tag{1.4}$$

We can now define the *wage-profit curve*. By substituting 1.3 into 1.4, we obtain the *wage-profit curve* associated with the set of methods $\mathbf{z}$:

$$w^{\mathbf{z}}(r, \eta) = [\eta' [\mathbf{B}^{\mathbf{z}} - \mathbf{A}^{\mathbf{z}}(1+r)]^{-1} \mathbf{L}^{\mathbf{z}}]^{-1} \tag{1.5}$$

Substituting 1.5 into 1.3 we obtain the price vector

$$\mathbf{p}^{\mathbf{z}}(r, \eta) = [\mathbf{B}^{\mathbf{z}} - \mathbf{A}^{\mathbf{z}}(1+r)]^{-1} \mathbf{L}^{\mathbf{z}} [\eta' [\mathbf{B}^{\mathbf{z}} - \mathbf{A}^{\mathbf{z}}(1+r)]^{-1} \mathbf{L}^{\mathbf{z}}]^{-1} \tag{1.6}$$

The price vector $\mathbf{p}^{\mathbf{z}}(r, \eta)$ is a function of the particular combination of methods $\mathbf{z}$, the profit rate r and the *numéraire*.

1.3 Wage-Profit Frontier and the Choice of Production Methods.

The outer envelope of all s possible[13] *wage-profit curves* is termed as the *wage-profit frontier*:

$$w_{\boldsymbol{E}}^{\text{WPF}}(r, \eta) = \max \left\{ w^{\mathbf{z}_1}(r, \eta), w^{\mathbf{z}_2}(r, \eta), ..., w^{\mathbf{z}_s}(r, \eta) \right\} \tag{1.7}$$

where $\boldsymbol{E}$ is a subset of $\boldsymbol{\Phi}$, ($\boldsymbol{E} \subset \boldsymbol{\Phi}$).

The domain of $w_{\boldsymbol{E}}^{\text{WPF}}(r, \eta)$ is composed of v intervals. The junctions between the different intervals are called *switch points* - points where the dominance of one *wage-profit curve* is replaced by another.

$$r \in \left[[0, \widehat{r}_1 [\cup [\widehat{r}_1, \widehat{r}_2 [\cup \ldots \cup [\widehat{r}_{q-2}, \widehat{r}_{q-1} [\cup [\widehat{r}_{q-1}, \widehat{r}_q [\cup [\widehat{r}_q, \widehat{r}_{q+1} [\cup \ldots \cup [\widehat{r}_{v-1} \mathcal{R}_{\boldsymbol{E}}^{WPF}] \right] \tag{1.8}$$

where $\widehat{r}_q$ $(q = 1, 2, \ldots, v-1)$ are the switch points and $\mathcal{R}_{\boldsymbol{E}}^{WPF}$ is the maximum rate of profit of $w_{\boldsymbol{E}}^{\text{WPF}}(r, \eta)$[14]. These intervals are relatively few with respect to the very large number of possible combination of methods[15].

Each interval, q, is the domain of a *wage-profit curve* that was generated by the set of methods $\mathbf{z}^{\{q\}}$. The whole set of methods that contribute to $w_{\boldsymbol{E}}^{\text{WPF}}(r, \eta)$ can be represented as a matrix:

$$\mathbf{Z}_{\boldsymbol{E}}^{\text{WPF}} = \left[\mathbf{z}^{\{1\}}, \mathbf{z}^{\{2\}}, \ldots, \mathbf{z}^{\{q\}}, \ldots, \mathbf{z}^{\{v\}} \right] = \begin{bmatrix} z_{11}^{\{1\}} & z_{12}^{\{2\}} & \ldots & z_{1q}^{\{q\}} & \ldots & z_{1v}^{\{v\}} \\ z_{21}^{\{1\}} & z_{22}^{\{2\}} & \ldots & z_{2q}^{\{q\}} & \ldots & z_{2v}^{\{v\}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z_{n1}^{\{1\}} & z_{n2}^{\{2\}} & \ldots & z_{nq}^{\{q\}} & \ldots & z_{nv}^{\{v\}} \end{bmatrix} \tag{1.9}$$

According to Samuelson, the above set of methods define the *reality* of *heterogeneous production* and the associated *wage-profit frontier*, eq. 1.7. It is this the reality that has to be approximated by the *fairy tale surrogate production function*. Whether this *fairy tale* *surrogate production function* has neoclassical properties depends on the set of methods.

1.4 National Accounting, Demand and the *Aggregate Production Function*

We now focus on the values associated with the combination of methods that determine the *wage-profit frontier*. This information is fundamental and useful to construct the efficient, aggregate production function.

It is known (see Bharadwaj (1970), Pasinetti (1977, Ch. 6) and Zambelli et al. (2014)) that the set of methods at the frontier, a combination, $\mathbf{z}^{\{q\}}$, will not change as the *numéraire* changes, but will change as a function of the profit rate. Furthermore, it is also known that two adjacent set of methods, $\mathbf{z}^{\{q\}}$ and $\mathbf{z}^{\{q+1\}}$, will differ only in one method. In other words, for a given profit rate r, there is a unique set of methods $\mathbf{z}^{\{q\}}$ that is associated to it. A change in the *numéraire* will not change the set of methods associated with r. It is only at switch points, $r = \widehat{r}_q$ that two different set of methods coexist: $\mathbf{z}^{\{q\}}$ and $\mathbf{z}^{\{q+1\}}$. Hence, the quantities produced at the *wage-profit frontier* are exclusively a function of the profit rate, r, and of the activity vector~~level of activity $\tilde{\mathbf{x}}_r$.~~ $\mathbf{x}$. In other words, for a given profit rate we have a unique determination of the combination of the frontier set of methods. This is an important property that simplifies the algorithmic complexity related to the computation of the aggregate values of the output and of capital.

In the following, we will work only with values computed at the frontier. In order to simplify notation we write $\tilde{\mathbf{z}}$ to mean the combinations of methods belonging to the frontier. That is $\tilde{\mathbf{z}} = \mathbf{z}^{\{q\}} \in \mathbf{Z}_{\boldsymbol{E}}^{\text{WPF}}$ with $q = 1, \ldots, v$.

The physical surplus and the net national product at the frontier is given by the vector [16]:

$$\mathbf{y}^{\tilde{\mathbf{z}}}(\mathbf{x}) = (\mathbf{B}^{\tilde{\mathbf{z}}} - \mathbf{A}^{\tilde{\mathbf{z}}})'\mathbf{x} \tag{1.10}$$

The value of aggregate net national product (measured using production prices) associated with a given profit rate r, combination of methods $\tilde{\mathbf{z}}$ and *numéraire* η is denoted as:

$$\mathcal{Y}^{\tilde{\mathbf{z}}}_{val}(r, \mathbf{x}, \eta) = \mathbf{x}'[\mathbf{B}^{\tilde{\mathbf{z}}} - \mathbf{A}^{\tilde{\mathbf{z}}}]\mathbf{p}^{\tilde{\mathbf{z}}}(r, \eta) = \mathbf{y}^{\tilde{\mathbf{z}}}(\mathbf{x})'\mathbf{p}^{\tilde{\mathbf{z}}}(r, \eta) \tag{1.11}$$

This is a scalar and is the aggregate value of output. Clearly even when the profit rate r and the *numéraire* η are given, as the activity vector changes the aggregate value of the net output will, almost always, change.

The physical capital is given by the vector:

$$\mathbf{k}^{\tilde{\mathbf{z}}}(\mathbf{x}) = \mathbf{A}^{\tilde{\mathbf{z}}\prime}\mathbf{x} \tag{1.12}$$

The value of capital (measured with production prices) is denoted as:

$$\mathcal{K}^{\tilde{\mathbf{z}}}_{val}(r, \mathbf{x}, \eta) = \mathbf{x}'\mathbf{A}^{\tilde{\mathbf{z}}}\mathbf{p}^{\tilde{\mathbf{z}}}(r, \eta) = \mathbf{k}^{\tilde{\mathbf{z}}}(\mathbf{x})'\mathbf{p}^{\tilde{\mathbf{z}}}(r, \eta) \tag{1.13}$$

We can call this measure as the *direct measurement of surrogate capital* (Samuelson, 1962, p.201). The total amount of employment is given by:

$$\mathcal{L}^{\tilde{\mathbf{z}}}(\mathbf{x}) = \mathbf{L}^{\tilde{\mathbf{z}}\prime}\mathbf{x} \tag{1.14}$$

Given the domain of the profit rate r and the *efficient* set of combinations, we have a unique triple $(\mathcal{Y}^{\tilde{\mathbf{z}}}_{val}(r,\mathbf{x},\eta), \mathcal{K}^{\tilde{\mathbf{z}}}_{val}(r,\mathbf{x},\eta), \mathcal{L}^{\tilde{\mathbf{z}}}(\mathbf{x}))$

Our objective is to *compute* the mapping $\mathcal{F}$ so that for a given couple ($\mathcal{K}^{\tilde{\mathbf{z}}}_{val}(r,\mathbf{x},\eta)$, $\mathcal{L}^{\tilde{\mathbf{z}}}(\mathbf{x})$), we can associate a unique value of $\mathcal{Y}^{\tilde{\mathbf{z}}}_{val}(r,\mathbf{x},\eta)$.

$$\mathcal{Y}^{\tilde{\mathbf{z}}}_{val}(r,\mathbf{x},\eta) = \mathcal{F}(\mathcal{K}^{\tilde{\mathbf{z}}}_{val}(r,\mathbf{x},\eta), \mathcal{L}^{\tilde{\mathbf{z}}}(\mathbf{x})) \tag{1.15}$$

Once a criteria for uniqueness has been assumed, we can claim to have *constructed* the aggregate production function[17].

A possible criterion could be to keep the surplus vector or physical net national product fixed, $\overline{\mathbf{y}}$. In this casefor a given profit rate r and the given efficient frontier methods $\tilde{\mathbf{z}}$, the activity level $\overline{\mathbf{x}}_r$ is uniquely determined and so are aggregate capital and output. There could be other criteria that are possible as well. For example, an alternative would be to pick, for a given profit rate, r, and *numéraire*, the activity level $\mathbf{x}^*_{r,\eta}$ which maximizes the value of output for a given value of capital (which is equivalent to the minimization of the value of capital for a given output: the isoquant). With one of these two criteria, we have a unique association between the value of the *surrogate capital* and the *surrogate output* or *surrogate net national product*. Clearly, the adoption of one criterion excludes the other.

1.5 The Neoclassical *as-if* or *surrogate* Production Function

The *aggregate production function* $\mathcal{F}$ defined in the previous section, eq. 1.15, may have different shapes. An important question concerns whether $\mathcal{F}$ is isomorphic with respect to the neoclassical production function(s) as defined in Appendix A. In order to be neoclassical, the computed *aggregate neoclassical production function* should have the following properties:

$$\frac{\Delta\mathcal{F}}{\Delta\mathcal{K}^{\tilde{\mathbf{z}}}_{val}} \geq 0; \frac{\Delta^2\mathcal{F}}{(\Delta\mathcal{K}^{\tilde{\mathbf{z}}}_{val})^2} \leq 0 \tag{1.16}$$

$$\frac{\Delta\mathcal{F}}{\Delta\mathcal{L}^{\tilde{\mathbf{z}}}} \geq 0; \frac{\Delta^2\mathcal{F}}{(\Delta\mathcal{L}^{\tilde{\mathbf{z}}})^2} \leq 0 \tag{1.17}$$

Δ is the difference operator. As we have explained in the introduction, the neoclassical postulate (Ferguson, 1969; Sato, 1974) requires that at the isoquants, the intensity of capital (i.e. the capital-labour ratio per unit of output) is negatively related to the profit rate and positively related to the wage rate. That is,

$$\Delta\left(\frac{\mathcal{K}^{\tilde{\mathbf{z}}}_{val}/\mathcal{L}^{\tilde{\mathbf{z}}}}{\mathcal{Y}^{\tilde{\mathbf{z}}}_{val}}\right)\Big/\Delta r \leq 0 \tag{1.18}$$

$$\Delta\left(\frac{\mathcal{K}^{\tilde{\mathbf{z}}}_{val}/\mathcal{L}^{\tilde{\mathbf{z}}}}{\mathcal{Y}^{\tilde{\mathbf{z}}}_{val}}\right)\Big/\Delta w \geq 0 \tag{1.19}$$

Following an established tradition we will call the points, or intervals, where the above do not hold, i.e. the points where $\left(\frac{\mathcal{K}^{\tilde{\mathbf{z}}}_{val}/\mathcal{L}^{\tilde{\mathbf{z}}}}{\mathcal{Y}^{\tilde{\mathbf{z}}}_{val}}\right)\Big/\Delta r > 0$, *Capital Reversal* or *Reverse Capital Deepening* points[18].

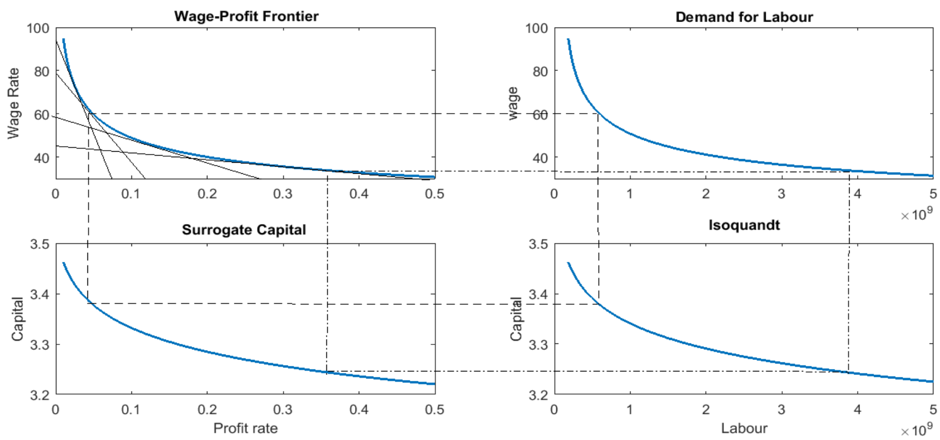

Figure 1.1: **Cobb-Douglas. Period 1995 to 2011**. $Y = AK^{\alpha}L^{1-\alpha}$. The parameters A and α are estimated with standard neoclassical procedures (Farrell, 1957). Once the values of A and α are given we have a direct computation of the Isoquant (south-east graph). The north-east graph is the implied *Neoclassical Demand for Labour*, eq. A.4. The south-west graph represents *the Demand for Capital* curve derived from eq. A.3. Subsequently the *wage-profit curve*, north-west, is computed from eq. A.4 and eq.A.3. The lines tangent to the *wage-profit curve* are hypothetical and represent Samuelson's view or conjecture. *[Data: See below sec. 3].*

Samuelson (1962) thought that the surrogate production function was neoclassical, implying that *capital reversing* phenomena would not occur for any profit rate. He assumed that the methods at the *wage-profit frontier* had a particular structure so that as the profit rate increases, the labour employed increases and the value of the *surrogate capital* decreases. This assumption was not tested.

Fig. 1.1 is an example of Samuelson's *as-if* or *surrogate* production function. The (virtual) straight lines in the north-west graph of Fig. 1.1 each represent a different set of methods $\mathbf{z}_j$, which in turn represent a *wage-profit relation* [19]. The north-west graph is qualitatively equivalent to the first two figures present in Samuelson (1962, p.195, p.197). It can be seen that there is an envelope forming the outer frontier of a large number of straight lines and that the envelope was convex to the origin. This convexity assumption was subsequently challenged from the theoretical point of view during the *Cambridge Capital* Controversy (see Garegnani (1966); Pasinetti (1966); Bruno et al. (1966)).

Samuelson had to admit that there was a theoretical possibility that the *surrogate production function* would not always hold neoclassical properties (Samuelson, 1966). But this did not shake the belief of many that the world is in fact neoclassical.

Until today, we do not find an actual algorithmic or empirical construction of the neoclassical surrogate production starting from heterogeneous production, as shown in subsections 1.3 and 1.4 above and which was originally indicated by Samuelson (1962) himself. Common practice is to assume the existence of an aggregate neoclassical production function where the isoquant (south-east graph of fig.1.1) is well behaved. Subsequently also the *wage-profit frontier* (north-weat graph of fig.1.1) is by construction well-behaved so as to exclude *capital reversing* by assumption.

In summary, the existence of a neoclassical aggregate production function is postulated but not empirically verified[20]. Graphical examples contradicting Samuelson conjecture encapsulated in fig. 1.1 of empirically verified *non neoclassical* cases are to be found below in figures 4.2 and 4.7.

2 Empirical Verification. General Setting.

In this section we provide a method to verify whether the aggregate unconditional production function (eq. 1.15) has neoclassical properties. This empirical verification requires several steps:

Step i) Defining the *book of blueprints* (eq. 1.1, the technological set Φ, see section 1.1);

Step ii) Computing of the *wage-profit curves*, as in eq. 1.5 and of the production prices, as in eq. 1.6 ;

Step iii) Determination of the efficient set of methods $\mathbf{Z}^{\text{WPF}}_{\boldsymbol{E}}$ and the computation of the associated *wage-profit frontier*;

Step iv) Computation of the *aggregate net national product* $\mathcal{Y}^{\tilde{\mathbf{z}}}_{val}(r, \mathbf{x}, \eta)$ (eq. 1.11) and of the aggregate *capital* $\mathcal{K}^{\tilde{\mathbf{z}}}_{val}(r, \mathbf{x}, \eta)$ (eq. 1.13). The computation of the values of these two magnitudes depend on the activity vector $\mathbf{x}$. In this paper and in the standard literature, the activity vector is determined in two different ways:

- **Isoproduct. Fixed Net National Product as a function of the profit rate**, $\overline{\mathbf{y}}$. The Physical Net National Product, or Surplus, is fixed a priori to be equal to $\overline{\mathbf{y}}$. Consequently, given the frontier set of methods, the activity level associated to this vector is uniquely determined by $\overline{\mathbf{x}}_r$. That is:

$$\overline{\mathbf{x}}_r = [(\mathbf{B}^{\tilde{\mathbf{z}}} - \mathbf{A}^{\tilde{\mathbf{z}}})']^{-1}\overline{\mathbf{y}} \qquad (2.1)$$

Given $\overline{\mathbf{x}}_r$ we can compute the unique tripple of aggregate values $\Big(\mathcal{Y}^{\tilde{\mathbf{z}}}_{val}(r,\overline{\mathbf{x}}_r,\eta),$ $\mathcal{K}^{\tilde{\mathbf{z}}}_{val}(r,\overline{\mathbf{x}}_r,\eta),\mathcal{L}^{\tilde{\mathbf{z}}}(\overline{\mathbf{x}}_r)\Big)$.

Recall that $\tilde{\mathbf{z}} = \mathbf{z}^{\{q\}} \in \mathbf{Z}^{\text{WPF}}_{\boldsymbol{E}}$ $(\boldsymbol{E} \subset \boldsymbol{\Phi})$ identifies one of the combination of methods at the frontier. There are v alternative combinations of efficient *wage-profit curves*, i.e. different methods producing $\overline{\mathbf{y}}$ belonging to the *wage-profit frontier*.

- **Isovalue. Fixed Value of Net National Product,** $\overline{\mathcal{Y}_{val}}$. The value of the Net National Product, or Surplus, is fixed *a priori* to be equal to $\overline{\mathcal{Y}_{val}}$. A standard criterium of efficiency is to produce a given value of the product at the lowest cost. There exists an activity vector $\mathbf{x}'$ which minimizes the value of the used physical means of production This minimum amount of aggregate value of capital $\mathcal{K}^{\tilde{\mathbf{z}}}_{val}(r,\mathbf{x},\eta)$ is associated with the unique activity vector $\mathbf{x}^*_{r,\eta}$ [21]. There are v different activity vectors associated to the intervals belonging to the frontier (see eq.1.8).

Step v) The final task or empirical verification is to check whether for a given *numéraire* η, the neoclassical postulate of eq. 1.18, i.e. $\left(\frac{\mathcal{K}^{\tilde{\mathbf{z}}}_{val}/\mathcal{L}^{\tilde{\mathbf{z}}}}{\mathcal{Y}^{\tilde{\mathbf{z}}}_{val}}\right)\Big/\Delta r \leq 0$ holds both for the aggregate system as well as for the individual industries.

It is quite surprising that the above empirical verification had not been conducted from the beginning to the end. In this paper, for the first time we follow all the five steps and attempt to close this research gap. Some explanation for this absence may be found in the fact that complete and comparable input-output tables were not available at the end of the 1960's. Leontief (1985) did attempt a first study on technological change with a *book of blueprints*, using the input output tables of the United States for 1979. In fact, the analysis was conducted on the *wage-profit curve* and not on the *wage-profit frontier*.

More complete dataset on input-output tables was assembled by the OECD, which started the project in the early 1990's and made the data available (for a limited set of countries) at the beginning of the 2000's. Han and Schefold (2006) used this data set to compute the *wage-profit curves* and did an analysis that compared *wage-profit curves* pairwise between countries, but never computed the global *wage-profit frontier* (eq.1.7) and hence never computed the set of the methods at the frontier, $\mathbf{Z}^{\text{WPF}}_{\boldsymbol{E}}$(eq.1.9).

We have not been able to find contributions where the *surrogate capital* and the *surrogate output* or *surrogate net national production* have been computed following Samuelson methods. An embryonic computation of the surrogate capital can be found in Leontief (1985). Ozol (1984) and Cekota (1988) computed a *surrogate wage function* with the use of few input-output tables, however in contexts where the aim was not to test the validity of the *surrogate production function*. They computed the *wage-profit curves* for Canada, but did not compute the surrogate capital and output. Attempts to check the tenability of the *surrogate production functions* have been attempted by Krelle (1977), Ochoa (1989), but, again, they worked only with a few *wage-profit curves*.

In short, all the empirical work we have found so far on the tenability of the *surrogate production function* have been mostly limited to the inferences on the presumed value of the *surrogate* or *jelly* capital derived by an indirect method. That is, from shape of the *wage-profit curves*, the shape of the capital-labour ratio was inferred (Samuelson, 1962, p.202), Harcourt (1972, p.143), Arrow et al. (1961, p.229) or Ferguson (1969, p.253). This may be justified by the lack of an algorithm that would allow the determination of the methods at the frontier.

In contrast, we utilize an algorithm that we have developed and presented in Zambelli et al. (2014) to compute the *wage-profit frontier* for the case of 31 sectors and 30 countries (and for 17 years) with total precision. We are in a position to compute the value of the *surrogate capital* following the direct method as in eq. 1.13. Once step iii) is fulfilled, the computation of the *surrogate production functions* is relatively straightforward.

3 Data and the Choice of the *Numéraires*

We use data from the World Input-Output Database (Timmer, 2012), a publicly available database that provides detailed input-output data at the industrial level for 35 industries from 1995-2011. The data set is composed of national input-output tables of 40 countries that includes 27 EU countries and 13 other major industrial countries. For more details regarding the construction of Input-Output tables in WIOD database, see Dietzenbacher et al. (2013).

We have confined our analysis to a subset of 30 countries[22]. Furthermore, we have reduced the total sectors or industries to 31. We are considering only those industries that belong to the core of the 'production' system [23]. The National Input-Output tables (NIOT) have been adjusted to include the imports of means of production. Hence, the methods associated with each sector would be the sum of inputs of internally produced goods and the inputs of the imported goods. All the current period values have been appropriately adjusted using the relevant price indexes. For this, we have used the data on price series available in the Social and Economic Accounts (SEA) section of the WIOD database (Timmer (2012)). The unique aspect of SEA is that it offers data at the industry level. Once the above adjustments have been made, we organize the means of production, labour inputs and the gross output as in the multidimensional matrix $\mathbf{\Phi}$. This enables us to enumerate all the possible combinations of methods of production with the vectors $\mathbf{z}$ and associate them to production systems formed by the triple: $\mathbf{A^z}$, $\mathbf{L^z}$, $\mathbf{B^z}$. We have used this information to compute the yearly *wage-profit frontier* and the set of methods associated, $\mathbf{Z}_{\boldsymbol{E}}^{\text{WPF}}$.

In order to obtain robust results, we have computed *wage-profit frontiers* and *surrogate production functions* relative to 32 different *numéraires* per year. These *numéraires* are the individual industries (Agriculture, Hunting, Forestry and Fishing – Mining and Quarrying . . . , and so on) and one which is a linear combination of all the industries.

Once the *numéraires* have been picked, the computations of the different values - *production prices*, *wage rates*, *capital-output ratios*, *capital intensity ratios* and so on - are carried out by assuming different *surpluses* or *net products*. These *surpluses* are divided into two broad categories used for the computations:

- **Isoproduct** We set the Net National Product, or physical surplus, to be equal only to one commodity, let us say i, while the production of the remaining $n-1$ commodities is only relative to the means of production necessary for the production of i[24]. An interesting case is when both the surplus and the *numéraires* are the same. Here, we reduce the measurement problem considerably. It is very convenient to study the properties of these production functions because there is only one output and not a composition of outputs and the *numéraire* is exactly that output. In this case, the notion of an isoquant is straight forward and if the *capital-labour ratio* per unit of output is not negatively related for the whole domain to the profit rate, the system is not *neoclassical*.

- **Isovalue** We fix the value of the Net National Product and determine the minimum quantity of the value of capital. This method is different from the Isoproduct method since the physical surplus changes its composition as the profit and the wage rates change due to changes in production prices and purchasing power. This is, in our view, more general than the previous method because a change in the distribution, the values of r, w and production prices, results in a change in the composition of the surplus.

In sum, the empirical investigation is conducted by constructing[25] the *surrogate production function* using the input-output tables relative to 17 years and 30 countries. The values of the aggregate output and the aggregate capital depend on the particular choice of the *numéraires* and on the composition of the surplus. For robust results, we have computed 576 aggregate production functions[26].

4 Results

4.1 Physical means of production.

The knowledge of the efficient set of methods facilitates a direct study of the n physical factors of production that are present as inputs in the input-output tables. The knowledge of the efficient set of methods and the distributional variables r and w together allow us to determine the inputs that the economic system as a whole shall use for the efficient production of a given surplus. As the frontier profit rates increase and the associated frontier wage rates decrease, the efficient set of methods and the physical quantities (the proportion between physical means of production, labour and output) associated with them change as well. A pure *neoclassical* case would be verified when, for given isoquant or physical surplus vector, the total labour input increases and the physical inputs decrease as the profit rate increases.

We fix the Net National Product to the vector $\bar{\mathbf{y}}$[27]. By fixing this, we can compute the ratio of *physical capital/output per employed* as a function of the decreasing rate of profit. That is, for each means of production i, we can compute its intensity.

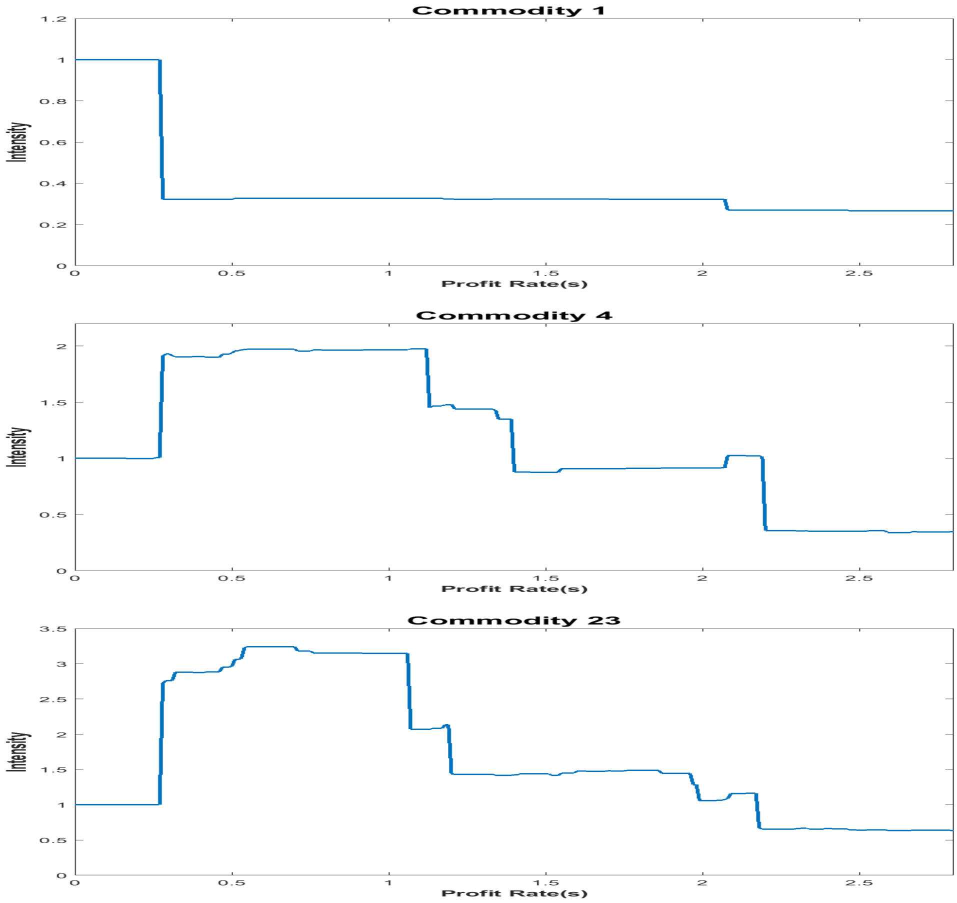

Figure 4.1: **Physical Capital Intensity per Commodity**. *Total means of production of individual commodities per labour as a function of the profit rate. Here the fixed surplus is in terms of Commodity 1, [*$\bar{\mathbf{y}}$*: Agriculture, Hunting, Forestry and Fishing]. and the technological set or the Book of Blueprints is relative to the years from 1995 to 2011. See table 4.1 for general results.*

Table 4.1: ***Intensity of the Physical Means of Production***. *Number of commodities for which the Capital Intensity*$(i, r, \overline{\mathbf{y}})$*, eq. 4.1, for a given Surplus* ($\overline{\mathbf{y}}$) *is negatively sloped for the whole domain* $r \in \left[0, \mathcal{R}_{\boldsymbol{E}}^{WPF}\right]$ *of the* wage profit rate frontier.

Fixed Net National Product ($\overline{\mathrm{Y}}$) – Individual Subsystems	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	1995-2011
1. Agriculture, Hunting, Forestry and Fishing	1	1	1	1	1	1	1	2	1	1	1	1	1	1	1	1	1	2
2. Mining and Quarrying	0	1	1	1	0	1	1	1	1	1	1	2	1	2	1	3	1	1
3. Food, Beverages and Tobacco	2	1	1	1	2	2	2	0	4	2	2	2	4	2	1	1	0	2
4. Textiles and Textile Products	0	1	1	1	1	0	0	0	3	2	1	1	1	1	1	1	0	0
5. Leather, Leather and Footwear	0	2	0	4	2	2	4	3	5	2	1	0	1	2	0	0	0	1
6. Wood and Products of Wood and Cork	0	1	0	0	0	1	1	0	2	1	0	2	1	1	1	0	1	1
7. Pulp, Paper, Paper , Printing and Publishing	1	1	1	1	0	1	1	1	2	1	2	3	2	2	1	2	1	1
8. Coke, Refined Petroleum and Nuclear Fuel	0	1	1	1	2	0	1	1	1	1	2	2	2	4	1	4	1	1
9. Chemicals and Chemical Products	1	1	1	3	1	1	1	1	3	1	1	1	1	1	2	1	1	1
10. Rubber and Plastics	0	1	0	0	0	0	1	1	0	0	1	3	1	1	2	1	2	2
11. Other Non-Metallic Mineral	1	0	1	0	0	1	0	1	0	1	1	1	1	1	2	2	2	3
12. Basic Metals and Fabricated Metal	1	2	1	2	1	1	1	1	2	1	1	1	1	1	2	1	1	1
13. Machinery, Nec	1	1	1	2	1	2	2	2	2	2	0	1	0	1	2	2	1	1
14. Electrical and Optical Equipment	1	1	1	2	0	2	2	2	3	0	2	2	4	1	2	2	1	1
15. Transport Equipment	1	1	1	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1
16. Manufacturing, Nec; Recycling	0	0	1	0	1	1	0	0	0	0	0	1	0	0	0	0	2	1
17. Electricity, Gas and Water Supply	0	0	2	0	1	0	0	2	1	1	1	1	1	1	1	4	2	2
18. Construction	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1
19. Sale, Maintenance and Repair of Motor Vehicles and Motorcycles; Retail Sale of Fuel	1	1	0	1	0	0	1	0	1	0	1	1	1	1	2	2	2	0
20. Wholesale Trade and Commission Trade, Except of Motor Vehicles and Motorcycles	0	2	1	2	1	1	0	0	3	2	2	2	3	1	2	2	2	0
21. Retail Trade, Except of Motor Vehicles and Motorcycles; Repair of Household Goods	1	0	1	0	1	0	1	0	0	1	0	0	1	0	1	0	0	0
22. Hotels and Restaurants	0	1	0	0	0	0	1	1	2	2	1	0	0	1	1	2	2	0
23. Inland Transport	1	1	1	1	1	1	1	2	2	1	1	1	0	1	1	1	1	0
24. Water Transport	1	0	0	0	0	0	1	0	1	1	1	1	1	1	1	1	1	0
25. Air Transport	2	0	1	1	0	1	0	1	2	2	0	0	0	2	0	0	0	3
26. Other Supporting and Auxiliary Transport Activities; Activities of Travel Agencies	1	0	1	1	0	0	0	0	0	0	1	1	1	1	1	1	1	1
27. Post and Telecommunications	0	2	1	0	1	1	1	1	3	1	1	2	2	5	2	2	2	1
28. Financial Intermediation	2	2	2	1	2	1	2	1	2	3	3	1	1	1	1	1	1	1
29. Real Estate Activities	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
30. Renting of M	1	1	4	3	3	0	0	0	3	3	1	3	2	2	1	2	2	1
31. Other Community, Social and Personal Services	1	1	1	0	1	1	1	0	1	2	1	0	1	1	2	1	1	0
32. Linear Combination of the Surplus of the 31 Subsystems	9	11	10	6	6	9	11	11	8	8	7	6	8	9	12	8	10	10

N.B: In the above table, each row represents the number of industries for which the neoclassical postulates hold, when the output of the industry (means of production) represented by the row is chosen as the surplus vector. The neoclassical postulates would require the means of production to be substituted with labour for all the 31 industries as the profit rate increases. The cell values in the table should be 31, if that were to be the case. However, we can see that there are only a few industries for which this holds. For example, let us consider the case when the surplus vector is *Agriculture, Hunting, Forestry and Fishing*. For a technological set or *Book of Blueprints* relative to the years from 1995 to 2011 (last column of the first row), only 2 out of 31 commodities exhibit capital intensity that is negatively sloped with respect to *r*. This example is shown in figure 4.1 .

$$\text{Capital Intensity}(i, r, \overline{\mathbf{y}}) = \frac{\text{Total Means of Production}(i, r, \overline{\mathbf{y}}) / \text{Total Production}(i, r, \overline{\mathbf{y}})}{\text{Total Labour Used}(i, r, \overline{\mathbf{Y}})} \tag{4.1}$$

This would be a step function of the profit rate (or wage rate). As the profit rate increases (and the wage rate decreases), the methods of production are the same as long as it belongs to the same interval (defined by two switch points, see eq. 1.8) and hence the total means of production used do not change. But as the profit rate increases and moves to another interval, the proportion of the means of production that are used change, although the net physical production vector is constant $\overline{\mathbf{y}}$. We should expect, following the *neoclassical* premises, that the quantity of labour employed increases (due to the fall in the wage rate) in substitution of the other means of production. This type of analysis does not require the computation of production prices and is independent of the choice of the *numéraire*: recall that the domain of the *wage-profit frontier*, the switch points and the total number of *wage-profit curves* are independent of prices[28].

The results are quite clear. We have computed the capital-labour ratios that are associated with the production of the individual inputs. That is, to the cases where the surplus is constituted of a single product and the remaining $n - 1$ inputs are produced just for the amount that is necessary. We have done these computations for each of the 17 years and for the whole period. The entries of the Table 4.1 are the number of means of productions for which as the profit rate increases there is substitution of physical means of production for labour. As we can see from the table, in the case of the single products we find that the number of means of production having this property is at most 3, out of a total of 31 means of production. In the case where the physical output is a composite commodity (last row of the table), we see that this number of increases to around 10, higher compared to 2 or 3, but still much below 31.

Figure 4.1 shows a sample (out of the 31) from the physical capital intensity function relative to the Agriculture subsystem for the whole period, from 1995 to 2011. This corresponds to the case represented by the 18^{th} value of the 1^{st} row of the table 4.1 . There are two commodities that behave in accordance with the *neoclassical postulate*. In the figure Commodity 1, Agriculture, is one of the two. The remaining 29 commodities have functions similar to the ones shown as Commodity 4 and 23. Clearly, as the wage rate decreases labour does become less expansive. Nevertheless, we do not find that all the physical means of production decrease in their use. The direction of change cannot be inferred from the increase (or decrease) of the profit rate r (wage rate w).

Strong evidence in favour of the tenability of the *neoclassical* production functions would have required that a substitution of all the physical means of production in favour of labour as the profit rate increases. In other words, the entries of Table 4.1 should have been equal to 31.

4.2 Isoproducts (direct method).

When we consider bundles of physical quantities, aggregation is not possible because the n commodities are distinct and *non-homogeneous*. If we sum tons of wheat with tons of iron, we would obtain a meaningless, spurious number. Aggregation might be meaningful only when using index-numbers or values.

We now focus our attention on the case where aggregate output and aggregate capital are computed with the index numbers *surrogate output*, eq.1.11, and, *surrogate capital*, eq. 1.13. We fix the physical surplus, the vector of produced goods, and compute the *surrogate capital* per total labour per unit of *surrogate output*, i.e., we compute the capital/output per labour intensity function, as in eq.1.18. We can call these cases where the physical net production is fixed as the *isoproduct* cases. It is simpler to analyse cases where the surplus produced and the *numéraire* are the same. In this way, the output is in fact a scalar, i.e., physical units of one good, and not a vector. The production prices are measured in terms of this surplus and hence the aggregate capital would be in value terms, homogeneous with the physical output.

We have computed values for a large number of couples of individual outputs (subsystems) and *numéraires*. Table 4.2 reports the statistics relative to the capital-output ratio per worker, all of which are computed across the years.

When the value is equal to 1, it indicates that the capital-output ratio per worker is consistent with the neoclassical case, i.e. the condition of eqs. 1.18 and 1.19 are verified for the whole domain, $r \in [0, \mathcal{R}_{\boldsymbol{E}}^{WPF}]$. Values below 1 would indicate the degree to which the function of the profit rate capital-output ratio per worker is negatively sloped. Values below 1 are clearly *not* consistent with the neoclassical postulate. For majority of cases (couples of fixed Net Product and *numéraire*) ranging around 85%, we find that the neoclassical postulate do not hold . Fig. 4.2 is one of these cases. In this example, the technological set is the total number of observations, the set Φ for the whole period going from 1995 to 2011.

The knowledge of $\mathbf{Z}_{\Phi}^{\text{WPF}}$ allows for the computation of the *wage-profit frontier*. For example, the outer envelope of all the possible combinations of methods observed during the period 1995-2011 is reproduced in the north-west graph of Fig. 4.2. It is computed based on the 100 curves that *dominate* all the other $31^{30x17} (\approx 3.6 \times 10^{760})$ possible *wage-profit curves*. The *wage-profit frontier* (north-west graph) and the *Demand for Labour at Isoquant* (north-east graph) are negatively sloped and this feature is independent of whether the set of methods $\mathbf{Z}_{\Phi}^{\text{WPF}}$ has neoclassical properties. These features would apply to any set of methods (for example see Sraffa (1960); Samuelson (1962)). Also, the *Demand for Labour at Isoquant* is negatively sloped with respect to the wage rate in most cases. But, this is to be expected: as the wage rate decreases, the most efficient methods of production might be those that utilize more labour. The neoclassical requirement is that as more labour is employed along the isoquant, less capital should be employed. This would give the standard negatively sloped, convex isoquant, which is the case that we would find in practically all microeconomics and macroeconomics text-books. An inspection of the south-west graph of fig. 4.2 makes it clear that this is not the case. At least, it is not the case for the whole domain, indicating that the economic system is not *neoclassical*.

Table 4.2: ***Isoproduct**. Capital-Output Ratios per Worker as a function of the profit rate, r for the case of a fixed physical surplus,* $\overline{\mathbf{y}}$ *(a vector). Measure of the neoclassical postulate (monotonicity).*

Numéraire (η) and Net Product $\overline{\mathrm{Y}}$ (Subsystem producing η) $\left(\overline{\mathrm{Y}}=\eta\right)$	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	1995-2011
1. Agriculture, Hunting, Forestry and Fishing	0.31	0.29	0.31	0.26	0.25	0.25	0.26	0.26	0.28	0.24	0.26	0.33	0.33	0.30	0.37	0.39	0.46	0.41
2. Mining and Quarrying	0.81	0.83	0.84	0.69	0.90	0.77	0.76	0.75	0.82	0.92	0.93	0.86	0.85	1	0.46	0.51	0.55	0.51
3. Food, Beverages and Tobacco	0.96	0.97	0.98	0.98	0.99	0.97	1	1	1	0.99	0.99	0.98	1	0.98	1	1	1	1
4. Textiles and Textile Products	0.96	1	1	0.96	0.95	1	0.98	1	1.00	1	0.96	1	1	0.98	0.82	0.80	0.81	0.89
5. Leather, Leather and Footwear	0.96	0.95	1	1	1	1	0.79	1	1	1	1	1	1	1	1	0.95	1	1
6. Wood and Products of Wood and Cork	0.98	0.99	0.97	0.95	0.92	0.95	0.95	0.97	0.98	0.96	0.94	0.95	0.92	0.92	0.97	0.94	0.91	0.96
7. Pulp, Paper, Paper , Printing and Publishing	1	0.96	0.86	0.96	1	0.90	0.96	0.90	0.97	0.97	0.96	0.92	0.95	0.94	0.88	0.84	0.87	0.86
8. Coke, Refined Petroleum and Nuclear Fuel	1	1	1	0.99	0.99	0.82	1	1	0.99	1	1	1	1	1	1	1	1	1
9. Chemicals and Chemical Products	1	1	1	1	1	1	1	1	1	0.90	0.89	0.90	0.87	0.96	0.90	0.82	0.77	0.79
10. Rubber and Plastics	1	1	1	1	1	1	0.96	0.94	0.98	0.97	0.99	0.98	1	0.97	1	1	0.96	0.97
11. Other Non-Metallic Mineral	0.84	0.89	0.88	0.93	0.95	0.91	0.96	0.97	0.97	0.92	0.91	0.91	0.89	0.96	0.99	0.94	0.92	0.93
12. Basic Metals and Fabricated Metal	0.89	0.86	0.77	0.99	0.89	0.94	0.90	1	0.99	1	0.98	0.94	0.93	1	0.89	0.87	0.83	0.77
13. Machinery, Nec	1	1	1	1	0.94	0.96	0.95	1	1	1.00	1	0.98	1	0.94	0.72	0.84	0.87	0.84
14. Electrical and Optical Equipment	1	1	0.99	0.97	1	1	1	1	1	1	1	1	1	1	0.90	0.77	0.77	0.91
15. Transport Equipment	0.71	0.67	0.59	0.66	0.75	0.70	0.78	0.83	1	1	0.99	1	0.95	0.93	0.78	0.82	0.87	0.95
16. Manufacturing, Nec; Recycling	0.75	0.65	0.77	0.74	0.77	0.76	0.77	0.86	0.77	0.68	0.81	0.95	0.92	0.94	1	1	1	1
17. Electricity, Gas and Water Supply	0.52	0.45	0.48	0.52	0.76	0.73	0.90	0.87	0.85	0.62	0.78	0.87	0.94	0.96	0.80	1	0.83	0.80
18. Construction	0.86	0.89	0.96	0.83	0.83	0.75	0.69	0.68	0.70	0.78	0.65	0.60	0.66	0.74	0.65	0.63	0.61	0.62
19. Sale, Maintenance and Repair of Motor Vehicles and Motorcycles; Retail Sale of Fuel	0.39	0.32	0.24	0.27	0.37	0.36	0.42	0.29	0.29	0.28	0.29	0.20	0.18	0.15	0.35	0.31	0.34	0.25
20. Wholesale Trade and Commission Trade, Except of Motor Vehicles and Motorcycles	0.23	0.23	0.28	0.21	0.26	0.28	0.28	0.26	0.16	0.15	0.18	0.23	0.22	0.16	0.13	0.19	0.18	0.18
21. Retail Trade, Except of Motor Vehicles and Motorcycles; Repair of Household Goods	0.17	0.13	0.15	0.12	0.12	0.14	0.12	0.12	0.12	0.15	0.15	0.18	0.16	0.13	0.13	0.13	0.13	0.15
22. Hotels and Restaurants	0.20	0.13	0.15	0.18	0.16	0.17	0.15	0.12	0.17	0.18	0.24	0.23	0.19	0.17	0.18	0.16	0.23	0.17
23. Inland Transport	0.43	0.42	0.47	0.38	0.38	0.30	0.29	0.29	0.43	0.46	0.52	0.47	0.56	0.47	0.51	0.52	0.52	0.37
24. Water Transport	0.73	0.65	0.76	0.66	0.82	0.88	0.90	1	0.92	1	1	0.99	0.99	0.96	0.97	0.98	0.99	0.99
25. Air Transport	0.87	0.88	0.82	0.83	0.88	0.86	0.93	0.62	0.52	0.95	1	1	1	1	1	1	1	0.89
26. Other Supporting and Auxiliary Transport Activities; Activities of Travel Agencies	0.49	0.55	0.44	0.42	0.52	0.50	0.46	0.37	0.44	0.32	0.28	0.32	0.37	0.35	0.32	0.30	0.42	0.49
27. Post and Telecommunications	0.33	0.40	0.58	0.63	0.38	0.54	0.58	0.17	0.49	0.41	0.33	0.22	0.31	0.18	0.23	0.33	0.37	0.33
28. Financial Intermediation	0.22	0.18	0.15	0.17	0.20	0.43	0.38	0.28	0.18	0.21	0.22	0.37	0.29	0.24	0.51	0.46	0.51	0.33
29. Real Estate Activities	0.98	1	0.94	0.97	1	0.98	0.96	0.83	0.98	0.85	0.86	0.84	0.86	0.90	0.90	0.88	0.88	0.93
30. Renting of M	0.32	0.34	0.36	0.24	0.27	0.36	0.32	0.32	0.24	0.28	0.32	0.36	0.37	0.25	0.36	0.35	0.35	0.40
31. Other Community, Social and Personal Services	0.33	0.22	0.28	0.26	0.24	0.27	0.23	0.20	0.18	0.18	0.20	0.17	0.16	0.16	0.21	0.20	0.29	0.31
32. Linear Combination of the Surplus of the 31 Subsystems	0.85	0.86	0.95	0.86	0.95	0.93	0.96	0.99	0.97	0.97	0.95	0.92	0.91	0.93	0.88	0.76	0.84	0.97

Each row is relative to the case in which the output is solely constituted by the production of that specific sector and the values are computed where *numéraire* is exactly the same output. The values are a measure of the degree to which the neoclassical postulate holds (see eq. 1.18). When the value is 1, it indicates that $\Delta\left(\frac{\mathcal{K}^{\tilde{\mathbf{z}}}_{val}/\mathcal{L}^{\tilde{\mathbf{z}}}}{y^{\tilde{\mathbf{z}}}_{val}}\right)\Big/\Delta r \leq 0$ for the whole domain of r. Values less than 1 indicate the degree (i.e. number of points over the total number of points) to which the function is negatively sloped.

For example, the case in which the Surplus is *Agriculture, Hunting, Forestry and Fishing*, for a technological set or *Book of Blueprints* relative to the years from 1995 to 2011 (last column of the first row), the cell value indicates that only 41% (0,41) of the points would be negatively sloped. This example is reported in figures 4.2 and 4.3.

A seemingly neoclassical case is that which relative to the third row (*Food, Beverages and Tobacco*), last column (*Book of Blueprints* relative to the period 1995-2011), where the value is 100% (1), see figures 4.4 and 4.5.

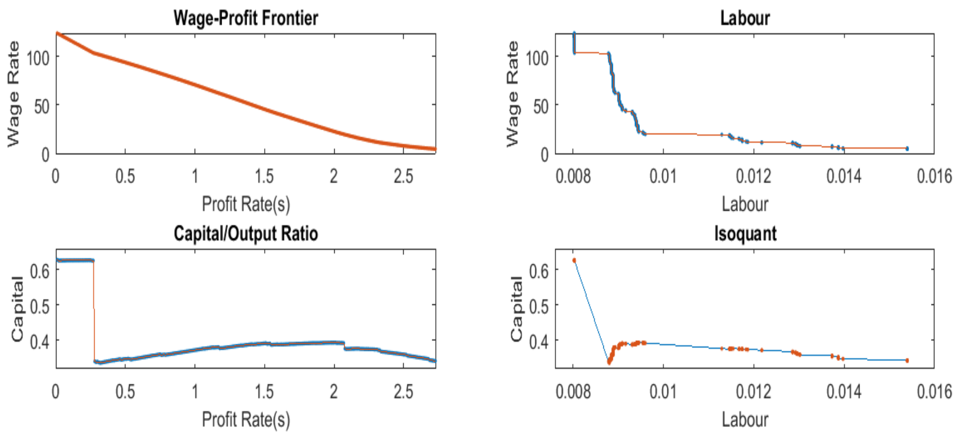

Figure 4.2: **Isoproduct or isoquant**. Technological set or Book of Bluprints relative to the period from 1995 to 2011.
Numéraire: Agriculture. Surplus: Agriculture Sector. The north-west graph is the *wage-profit frontier*, which is the envelope of the *wage-profit curves*. The north-east graph, *The Demand for labour at Isoquant* is the quantity of labour necessary for the production of the same value quantity of the fixed surplus vector. The south-west graph is *the Demand for Capital* curve The south-east graph is the *isoquant* or *isoproduct* curve. For general results related with the isovalues see table 4.2. This figure is relative to the entry first row last column of table , 0.41 or 41%.

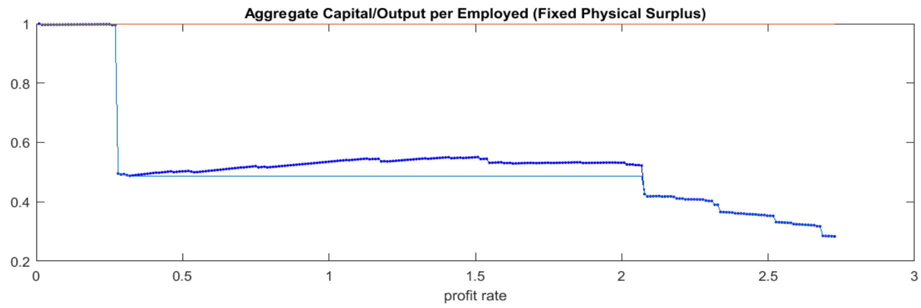

Figure 4.3: Years 1995-2011. Isoproduct. Aggregate Capital-Output ratio per worker (Surplus of the Agriculture Sector and Agriculture as *numéraire*): $\frac{\mathcal{K}^{\tilde{\mathbf{z}}}_{val}/\mathcal{L}^{\tilde{\mathbf{z}}}}{\mathcal{Y}^{\tilde{\mathbf{z}}}_{val}}$ *. The point s above the thin line are to be considered* non-neoclassical. *Same data used for 4.2*

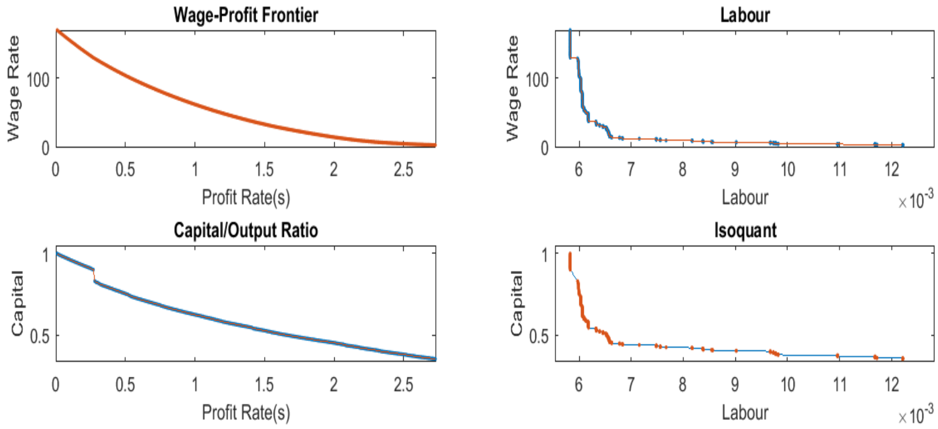

Figure 4.4: Aggregate Values of an Heterogeneous Production System for the case of Physical Net National Production (*Numéraire* and Surplus relative to Industry 3: Food, Beverages and Tobacco). *See fig. 4.2 for a detailed description of the graphs. [Technological set or Book of Blueprints relative to the years from 1995 to 2011].*

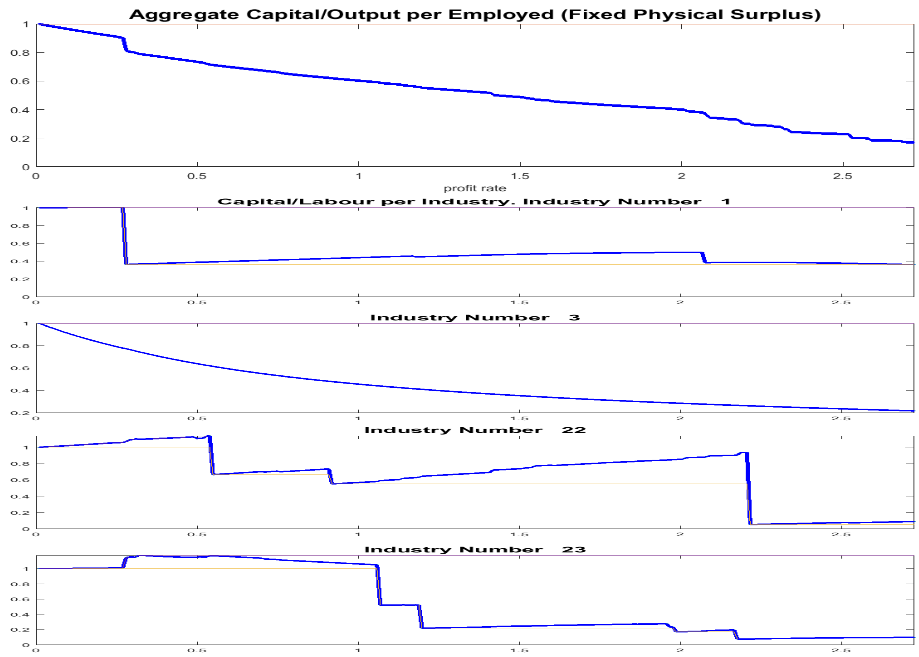

Figure 4.5: Aggregate Capital/Ouput per worker, top graph, and a sample of the individual industry capital/output per worker (*Numéraire* and fix surplus of Industry 3: Food, Beverages and Tobacco). *[Technological set or Book of Blueprints relative to the years from 1995 to 2011].*

At first glance, a scholar trained to think in terms of *as-if* marginal productivities and substitution among factors may find the positive relation in the isoquants between labour and capital disturbing or even counter intuitive. Although this may be an unpleasant result, it is an actual possibility and in fact, it is the normal case. As the profit and wage rates change, there is a change in accounting equilibrium prices. Eventually and/or consequently, there would also be a change in the most efficient methods of production and a change in the measure of the aggregates. Although, in this case, the physical net output produced is the same.

Figure 4.3 is the aggregate capital-output ratio per worker. We can see that there is a sizeable domain of the curve where the neoclassical relations, eqns. 1.18 1.19 above, do not hold. There are instances, couples of fixed surplus and *numéraire*, in which the aggregate capital-labour per unit of output intensity values are negatively sloped for the whole domain of the profit rate. These cases are the instances in Table 4.2 where the value is equal 1.

It is worth pointing out that the negative slope is a *necessary* condition for the whole system to be considered neoclassical, but it is *not sufficient*. In the input-output table, n goods are produced and hence we have n production functions, producing the n physical factors or means of production. These production functions would be *neoclassical* if the capital-output ratio per worker would also be negatively sloped with respect to the changes in the profit rate (see the appendix A, eq. A.27).

In all our computations we have not found a single case where all the industries are *neoclassical* and it has been quite the contrary. At the individual industry level, we find that all the instances are qualitatively the same. That is, the cases which appear to be *neoclasssical* at the aggregate level are not distinguishable from the *non neoclasssical* cases.

Figure 4.4 is a good example of a system seems to be *neoclassical* in the aggregate, but the individual industry level curves are not. The curves of the four graphs all have *neoclassical properties* taken as a sample of the 31 curves, each relative to a commodity or industry. This would be the case associated to the third row and the last column of 4.3, i.e. *numéraire* and surplus of commodity-industry 3 relative to the whole period, 1995-2011. When we analyse the capital/intensity of the use of the individual means of production, we see that the seemingly neoclassical cases are not different from the non-neoclassical ones. Figure 4.5 reports a sample of the industry level capital-output ratio per worker. The top graph is the aggregate capital-output ratio per worker. Being negatively sloped, it could be consistent with the neoclassical postulate. However, an inspection of the sample of industry-level curves shows that only the production of the third industry is consistent with the *neoclassical* premise. Only the value of capital-output ratio per worker of industry 3 is negatively sloped for the whole domain.

As the *numéraires* change, the values of the capital-output ratios per worker change as a function of the profit rate (even when the physical production does not change) and it is in a direction that is contrary with the neoclassical postulates. Table 4.3 reports the number of industries that would have more than 90% of the points negatively sloped, the remaining ones have lower values. The number of industries change as the *numéraires* change. Similar tables as Table 4.3 have been computed for different thresholds. For reasons of space we do not publish all these tables. We find that the results are quite negative. None of the industries are 100% neoclassical for a majority of the instances. In Table 4.3 the numbers associated with a * are relative to the case where there is only one industry that is 100% neoclassical.

Table 4.3: ***Isoproduct**. Number of Industries where the Capital-Output Ratio per Worker is negatively sloped for more than 90% of the domain in r (The total number of industries is 31)*

Numéraire (η) and Net Product $\overline{Y}$ (Subsystem producing η) $(\overline{Y} = \eta)$	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	1995-2011
1. Agriculture, Hunting, Forestry and Fishing	5	5	7	5	4	4	7	8	11	8	8	10	8	10	8	6	5	7
2. Mining and Quarrying	3	6	4	5	3	1	6	10	13	10	7	8	9	9*	8	6	6	8
3. Food, Beverages and Tobacco	5	5	5	8	6	2	4	8	13	7	8	8	10	11	8	6	6	8*
4. Textiles and Textile Products	5*	9*	8*	7	7	4*	6	9*	11*	8*	10	8*	10*	10	7	6	6	8
5. Leather, Leather and Footwear	6	8	7*	8*	5*	4*	4	8*	12*	9	9	8	10*	10*	7	7	5	6
6. Wood and Products of Wood and Cork	4	5	7	7	6	3	7	9	12	7	8	9	11	9	9	7	7	7
7. Pulp, Paper, Paper , Printing and Publishing	5*	10	9	6	7	6	8	10	13	9	10	10	10	11	7	6	6	8
8. Coke, Refined Petroleum and Nuclear Fuel	3*	5*	6*	5	4*	4	6*	9*	8*	6*	7	8	9	10	8*	6	6*	8
9. Chemicals and Chemical Products	5*	7*	8*	7*	6*	2*	6*	8*	11*	8	8	8	9	12	7	6	6	7
10. Rubber and Plastics	5*	10*	7*	7	5*	5*	9	9	14	11	10	8	10	12	9*	6	6	8
11. Other Non-Metallic Mineral	4	4	7	5	5	2	7	9	12	11	8	7	8	10	8	7	6	9
12. Basic Metals and Fabricated Metal	4	6	7	7*	3	3	5	9	10	8	10	9	9	10	8	6	5	8
13. Machinery, Nec	5*	7*	8*	7*	7	5	6	10*	14*	11*	10	10	9	12	9	6	6	8
14. Electrical and Optical Equipment	6*	7*	7	6	5	5*	6*	8*	13*	10	10*	9*	10*	13*	9	6	6	8
15. Transport Equipment	4	8	7	6	5	5	5	10	14	9	11	11	9	11	9	6	6	9
16. Manufacturing, Nec; Recycling	6	7	9	7	6	7	7	10	14	9	11	8	10	12	9*	8*	8	9*
17. Electricity, Gas and Water Supply	4	5	7	4	5	3	6	10	8	8	6	8	9	10	8	7	5	8
18. Construction	3	8	7	5	4	3	6	9	12	8	8	9	9	11	9	8	7	8
19. Sale, Maintenance and Repair of Motor Vehicles and Motorcycles; Retail Sale of Fuel	4	8	7	7	5	2	6	8	12	8	9	9	8	10	7	6	6	7
20. Wholesale Trade and Commission Trade, Except of Motor Vehicles and Motorcycles	4	9	7	5	6	7	8	7	11	8	8	9	9	12	7	6	6	9
21. Retail Trade, Except of Motor Vehicles and Motorcycles; Repair of Household Goods	4	8	8	6	6	5	5	7	11	8	7	8	10	10	7	6	5	9
22. Hotels and Restaurants	5	7	8	6	5	4	5	6	9	6	5	7	10	9	8	6	5	9
23. Inland Transport	3	5	6	6	6	2	6	8	9	6	8	8	9	10	7	6	6	8
24. Water Transport	4	4	8	6	5	3	6	7	10	7	8	8*	9*	10*	6*	6*	5*	7
25. Air Transport	3	7*	6	6	6	3	4	6	10	10	7*	8*	9*	9	6	5	5	9
26. Other Supporting and Auxiliary Transport Activities; Activities of Travel Agencies	3	8	6	4	5	2	5	6	12	7	7	9	10	10	7	6	6	9
27. Post and Telecommunications	4	6	7	6	6	3	7	6	8	7	6	7	10	10	8	6	6	7
28. Financial Intermediation	5	7	9	6	6	4	7	7	8	7	7	7	10	11	7	6	5	8
29. Real Estate Activities	3	5	8	4	4	4	7	8	11	6	7	8	10	11	8	5	5	9
30. Renting of M	6	6	8	7	8	4	7	8	11	9	7	7	10	9	7	6	5	9
31. Other Community, Social and Personal Services	6	8	7	6	6	5	9	9	10	8	8	9	9	11	7	6	5	8
32. Linear Combination of the *numéraires* (and Subsystems)	7	9	9	6	7	5	7	11	12	10	8	9	10	11*	9	8	6	8*

Each row is the number of industries for which the the industry intensity (capital-output ratio per worker) has at least 90% of the points relative to the domain r which are negatively sloped - i.e. for more than 90% of the points we observe that $\Delta\left(\frac{\mathcal{K}^{\tilde{z}}_{val}/\mathcal{L}^{\tilde{z}}}{\mathcal{Y}^{\tilde{z}}_{val}}\right)\Big/\Delta r \leq 0$.

* The entries with the * are relative to the number of cases ~~industries~~ that would be 100% neoclassical i.e where the aggregate Capital-Output Ratio per worker is negative sloped for the whole domain These cases have in table 4.2 value 1.These are cases where the value of the capital intensity of one industry which is negative sloped for the whole domain dominates all the values of the remaining industries. There are only a few instances where there is only one industry, none with more than one (as an example see figures 4.4 and 4.5.), while the majority of the instances would not have cases where there is at least one industry for which the capital intensity is negative for the whole domain.

Figure 4.6 summarizes the results for the capital-output ratio per worker for the n industries. We take the average[29] of the number of industries that would be totally neoclassical in the sense that the capital-output ratio per worker is negatively sloped for the whole domain (Degree of Membership equal to 1). The average is very low, 0.146. This means that for the majority of the cases there is, on average, not even one industry (out of 31) that would have a *well-behaved* capital-output ratio per worker

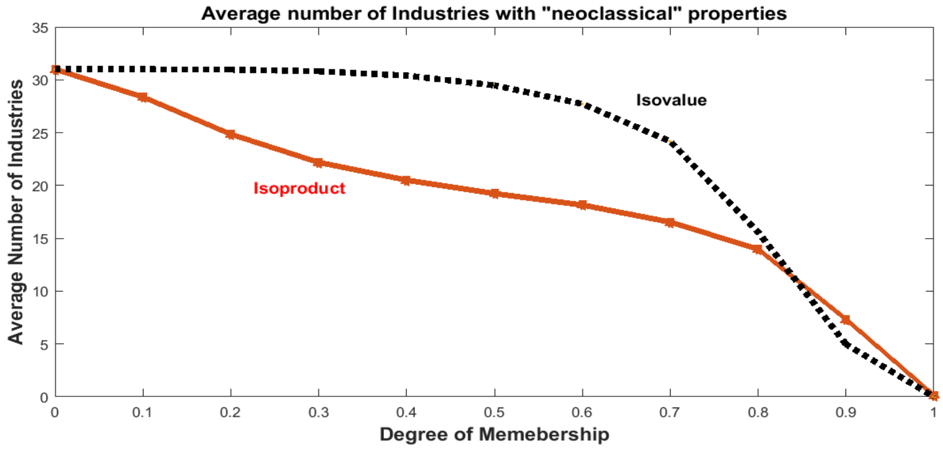

Figure 4.6: *Average Number of Industries having neoclassical properties classified by the degree of membership. Value number 1 is the case where the capital-output ratio per worker per industry is negatively sloped - w.r.t. the profit rate - for the whole domain. Value number 0.9 would be the case where the capital-output ratio per worker is negatively sloped for 90% of the points (This is the mean of the values of Table 4.3 for the case of the isoproduct). The average is computed by considering all the instances (the numéraires) and all the years.*

curve. Clearly, strong evidence in favour of the *neoclassical* case would have required that the average is 31 throughout the whole Degree of Membership. Obviously, this is not the case.

4.3 Isovalues (direct method).

In the previous section, we have analysed the instances where the isoquants are computed by keeping the physical surplus fixed, but with changing profit rate and wage rate. Here we keep the value of net national product fixed, $\overline{\mathcal{Y}}^{\tilde{\mathbf{z}}}_{val}$ (see eqs. A.29–A.32) and compute the associated isoquants. The computations of the isoquants with fixed physical surplus (isoproduct) as the profit rate and wage rate change does have the advantage of being of a straight forward interpretation. But one has to assume that as the profit rate, the wage rate, the production prices and distribution change, the array of demanded goods that form the surplus does not change at all.

Here, we keep the value of the Net National Product fixed and compute the associated minimum value for capital allowing for the surplus vector to adjust accordingly. When we generate aggregate values relative to all the years or to the whole period going from 1995 to 2011, we keep the *numéraire* fixed. Therefore, all the computed aggregates are in terms of the same bundle of goods, i.e. the purchasing power of the same bundle of physical value. This is very important. We cannot know what the effective demand is and how it would change as the profit and wage rates change , but we can determine an efficient relation in value. That is, the minimum value of capital necessary for the whole system to produce the fixed value of the aggregate output,

Table 4.4: ***Isovalue**. Capital-Output Ratios per Worker as a function of the profit rate, r for the case of a fixed aggregate value of the surplus* $\mathcal{Y}^{\tilde{\mathbf{z}}}_{val}$ *(a scalar), eq. 1.11. Measure of the neoclassical postulate (monotonicity).*

Numéraire (η)	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	1995-2011
1. Agriculture, Hunting, Forestry and Fishing	0.58	0.61	0.76	0.58	0.55	0.67	0.69	0.71	0.41	0.38	0.40	0.47	0.58	0.41	0.45	0.43	0.75	0.58
2. Mining and Quarrying	0.74	0.83	0.78	0.71	0.88	0.83	0.70	0.75	0.89	0.93	0.88	0.85	0.84	0.98	0.78	0.80	0.54	0.61
3. Food, Beverages and Tobacco	0.96	0.91	0.98	0.91	0.95	0.97	0.96	0.98	0.99	0.99	0.95	0.98	0.97	0.96	0.98	0.98	0.98	0.95
4. Textiles and Textile Products	0.99	1.00	1.00	0.99	0.98	0.93	0.91	1.00	0.96	0.99	0.91	0.92	0.91	0.88	0.80	0.80	0.75	0.82
5. Leather, Leather and Footwear	0.98	0.94	0.90	1.00	0.98	0.99	0.91	1.00	0.97	0.98	0.98	0.99	0.99	0.99	0.92	0.85	0.98	0.96
6. Wood and Products of Wood and Cork	0.95	0.98	0.97	0.89	0.92	0.97	0.96	0.98	0.97	0.96	0.92	0.94	0.92	0.90	0.96	0.92	0.89	0.94
7. Pulp, Paper, Paper , Printing and Publishing	0.99	0.94	0.86	0.96	0.98	0.89	0.96	0.88	0.97	0.95	0.94	0.90	0.94	0.93	0.86	0.78	0.86	0.85
8. Coke, Refined Petroleum and Nuclear Fuel	0.88	0.95	0.98	0.97	0.95	0.81	0.98	0.97	0.97	0.97	0.98	0.99	0.99	0.98	0.99	0.98	0.98	0.98
9. Chemicals and Chemical Products	1.00	1.00	1.00	0.98	0.98	0.97	0.98	1.00	1.00	0.88	0.88	0.91	0.85	0.97	0.89	0.80	0.77	0.79
10. Rubber and Plastics	1.00	1.00	1.00	1.00	0.98	1.00	0.96	0.94	0.99	0.90	0.93	0.96	0.92	0.96	0.98	0.87	0.86	0.95
11. Other Non-Metallic Mineral	0.88	0.89	0.87	0.96	0.95	0.95	0.98	0.97	0.99	0.96	0.95	0.92	0.92	0.98	0.97	0.95	0.92	0.92
12. Basic Metals and Fabricated Metal	0.89	0.84	0.76	0.98	0.88	0.93	0.89	1.00	0.99	0.99	0.97	0.97	0.91	0.98	0.92	0.91	0.89	0.80
13. Machinery, Nec	0.93	0.96	0.92	1.00	0.94	0.94	0.92	1.00	1.00	1.00	0.98	0.98	0.95	0.97	0.85	0.88	0.93	0.91
14. Electrical and Optical Equipment	0.99	0.95	0.91	0.99	0.98	1.00	0.98	1.00	1.00	0.99	0.99	0.99	0.98	0.98	0.95	0.79	0.89	0.97
15. Transport Equipment	0.71	0.64	0.54	0.71	0.68	0.74	0.85	0.97	1.00	1.00	0.98	0.98	0.96	0.95	0.87	0.81	0.86	0.87
16. Manufacturing, Nec; Recycling	0.91	0.69	0.71	0.82	0.67	0.82	0.74	0.86	0.81	0.90	0.95	0.97	0.96	0.95	0.97	0.91	0.99	0.96
17. Electricity, Gas and Water Supply	0.94	0.82	0.85	0.87	0.91	0.82	0.82	0.98	0.95	0.94	0.96	0.93	0.93	0.96	0.94	0.75	0.91	0.71
18. Construction	0.96	0.98	0.99	0.92	0.95	0.94	0.96	0.90	0.89	0.91	0.86	0.83	0.85	0.89	0.86	0.81	0.81	0.82
19. Sale, Maintenance and Repair of Motor Vehicles and Motorcycles; Retail Sale of Fuel	0.40	0.37	0.38	0.60	0.58	0.61	0.50	0.56	0.50	0.53	0.52	0.50	0.48	0.48	0.53	0.41	0.56	0.52
20. Wholesale Trade and Commission Trade, Except of Motor Vehicles and Motorcycles	0.39	0.17	0.20	0.30	0.36	0.46	0.20	0.22	0.26	0.18	0.18	0.23	0.22	0.17	0.17	0.21	0.21	0.20
21. Retail Trade, Except of Motor Vehicles and Motorcycles; Repair of Household Goods	0.19	0.14	0.16	0.15	0.21	0.18	0.15	0.14	0.14	0.15	0.15	0.16	0.17	0.14	0.15	0.17	0.20	0.16
22. Hotels and Restaurants	0.59	0.46	0.41	0.40	0.40	0.37	0.22	0.30	0.28	0.31	0.28	0.29	0.39	0.33	0.36	0.32	0.42	0.28
23. Inland Transport	0.44	0.47	0.47	0.55	0.52	0.35	0.46	0.44	0.52	0.52	0.49	0.52	0.56	0.53	0.55	0.55	0.50	0.47
24. Water Transport	0.66	0.64	0.66	0.66	0.80	0.86	0.86	0.90	0.90	0.95	0.88	0.94	0.86	0.70	0.79	0.71	0.83	0.76
25. Air Transport	0.85	0.74	0.77	0.81	0.83	0.70	0.81	0.59	0.56	0.76	0.97	0.85	0.91	0.75	0.85	0.64	0.86	0.81
26. Other Supporting and Auxiliary Transport Activities; Activities of Travel Agencies	0.52	0.56	0.42	0.50	0.60	0.58	0.49	0.49	0.50	0.35	0.32	0.38	0.41	0.43	0.50	0.50	0.51	0.59
27. Post and Telecommunications	0.34	0.30	0.46	0.61	0.62	0.69	0.50	0.31	0.33	0.42	0.36	0.22	0.26	0.22	0.21	0.31	0.30	0.17
28. Financial Intermediation	0.25	0.18	0.15	0.18	0.23	0.27	0.20	0.22	0.19	0.22	0.22	0.24	0.32	0.32	0.44	0.40	0.51	0.32
29. Real Estate Activities	0.99	0.98	0.90	0.96	0.96	0.94	0.95	0.82	0.95	0.82	0.85	0.80	0.83	0.88	0.90	0.86	0.87	0.81
30. Renting of M	0.30	0.26	0.22	0.39	0.38	0.43	0.25	0.25	0.28	0.27	0.25	0.31	0.35	0.25	0.47	0.49	0.58	0.39
31. Other Community, Social and Personal Services	0.31	0.34	0.40	0.40	0.37	0.53	0.20	0.23	0.20	0.20	0.18	0.21	0.23	0.22	0.27	0.27	0.42	0.31
32. Linear Combination of the Surplus of the 31 Subsystems	0.74	0.76	0.65	0.80	0.77	0.76	0.77	0.86	0.87	0.82	0.88	0.84	0.89	0.84	0.82	0.76	0.71	0.79

Each row is relative to the case in which the output is constituted only of the production of that specific sector and the values are computed having as *neméé*raire exactly the same output. The values are a measure of the degree in which the neoclassical postulate holds (see eq. 1.18). The value 1 indicates that it is the case that $\Delta\left(\frac{\mathcal{K}^{\tilde{\mathbf{z}}}_{val}/\mathcal{L}^{\tilde{\mathbf{z}}}}{\mathcal{Y}^{\tilde{\mathbf{z}}}_{val}}\right)\Big/\Delta r \leq 0$ for the whole domain of r. Values less of 1 indicate the degree (i.e. number of points over the total number of points) in which the function is negatively sloped. Here as the profit rate changes the value of the surplus $\mathcal{Y}^{\tilde{\mathbf{z}}}_{val}$ is kept fixed (and the value of capital is minimized). This table differs from table 4.2, Here the the vector of physical quantities $\overline{\mathbf{Y}}$ is variable and the value of the aggregate surplus $\mathcal{Y}^{\tilde{\mathbf{z}}}_{val}$ is kept fixed as the profit rate changes. The opposite happens in table 4.2.
For example, the case in which the *numéraire* (η) is *Agriculture, Hunting, Forestry and Fishing*, for a technological set or *Book of Blueprints* relative to the years from 1995 to 2011 (last column of the first row), the cell value would indicate that only 58% (0.58) of the points would be negatively sloped. This example is reported in figure 4.7.

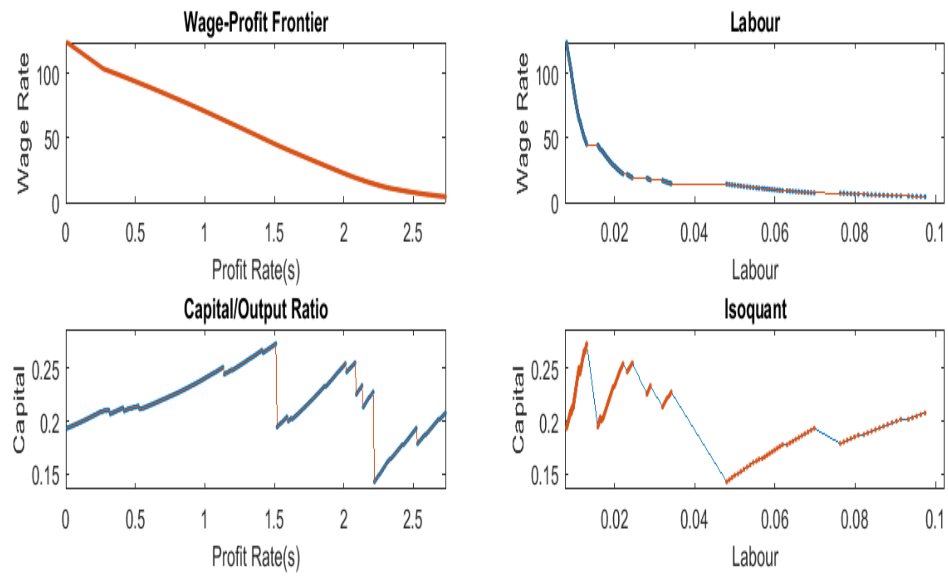

Figure 4.7: **Isovalue or Isoquant.**. Technological set or Book of Blueprints relative to the period from 1995 to 2011.
Numéraire: Agriculture.
The north-west graph is the *wage-profit frontier*, which is the envelope of the *wage-profit curves*. The north-east graph, *The Demand for labour at Isoquant* is the quantity of labour necessary for the production of the same value quantity of the surplus vector. The south-west graph is *the Demand for Capital* curve The south-east graph is the *isoquant* or *isovalue* curve. See fig. 4.2 for a detailed description of the meaning of the four graphs. For general results related with the isovalues see table 4.4. This figure is relative to the entry first row last column of table 4.4, 0.58 or 58%.

$\overline{y}^{\tilde{\mathbf{z}}}_{val}$.

In table 4.4, we have generated the same statistics, and tables equivalent to the above tables 4.2 4.3, but now for the case of isovalues. We can see that the results are qualitatively the same as for the case of the isoproducts. Figure 4.7 is generated using the same data as in figure 4.2. However, the isoproduct is replaced with the isovalue curve in the latter. A comparison between these two graphs shows that the isoquants (South-East graph) are even more problematic from a neoclassical point of view in this case.

4.4 *'Fairy tale'* production function and capital reversing (indirect method).

In the previous section, we have computed the value of aggregate capital using a direct method. That is, we have computed the value of capital and the capital-output per employed ratio using quantities and production prices. As we have explained above (see in particular section 2), this is the first time that this has been done to the best of our knowledge. The contribution of Zambelli et al. (2014) played a crucial role in enabling us to compute the methods of production associated to the *wage-profit frontier* with precision and for non trivial cases. This knowledge allows a precise computation of the frontier wage rate, profit rate and production prices. For the sake of completeness, we present the results for the case in which the value of capital and hence the value of the capital-labour ratio are inferred indirectly from the knowledge of the *wage-profit frontier*. Inspired by a definition given by Samuelson, we will call this capital, *fairy tale* (see the quote above pag. 5).

In the Appendix we have shown, following standard neoclassical assumptions and the literature (eq. A.16), that it must be the case that :

$$\frac{dw}{dr} = -k \tag{4.2}$$

where k is the capital-labour ratio or *fairy tale* capital[30].

From the shape of the *wage profit frontier* one could infer whether *surrogate capital* could exhibit *capital reversing* or *reswitching*[31] by observing the convexity or concavity of the *wage-profit curve* (or the monotonicity of prices). If the *wage-profit curve* is concave to the origin, this would imply that the *surrogate capital* or *"jelly" capital* exhibits *capital reversing*, i.e. the *surrogate production function* is not neoclassical. This means that the functional forms postulated (see below Appendix A, p.31) are not, using Samuelson's own words, verified by *reality* and the neoclassical *fairy tale* does not hold. On the contrary, if *wage-profit curve* is convex to the origin or if it is linear, the possibility of *capital reversing* could be excluded and hence the *surrogate production function* could have neoclassical properties. This was at the center of the two Cambridges debate. A good exposition is to be found in (Harcourt, 1972, p.40-45 and pp.122-130), Ferguson (1969, pp.251-53) or Felipe and McCombie (2013).

It has to be pointed out that the convexity or linearity of the *wage-profit frontier* is only a *necessary* condition for the production system to be classified as *neoclassical*, but it is not a *sufficient* condition. The explanation is similar to the case discussed above, section 4.2, fig.4.4 and fig.4.5 [32].

Recent work by Shaikh (1988), Han and Schefold (2006), Mariolis and Tsoulfidis (2011), Shaikh (2012) and Schefold (2013a) provide some evidence of the empirical existence of *quasi-linear wage profit frontier*. Supported by this finding, they claim that they can exclude, in general, *capital reversing* phenomena. In the case of absolute linearity, dw/dr is equal to a constant, hence the capital-labour ratio is not positively related with the *profit rate* and the *neoclassical* postulate holds.

The empirical work of these authors is based on different data sets. They compute the shape - i.e. slope - of the *wage-profit curves* of individual countries (Canada, United States and Germany), but do not present or work on explicit computations of the *surrogate capital*. Han and Schefold (2006) make pairwise comparisons between the *wage-profit curves* of two countries. Though we do not engage in a thorough discussion of their contributions, it is our view that their claim that *capital reversal* can be excluded is problematic for several reasons.

There is a logical reason that concerns the very concept of *quasi-linearity* of the *wage-profit curve*. If the *wage-profit frontier* is *quasi-linear*, it implies that it is not completely linear. Hence, it is either concave or convex. This does not exclude the concavity of the curve, and it is quite the contrary. Another problem concerns the fact that the convexity or concavity of the *wage-profit curves* depend on the *numéraire*. Therefore, a curve which is convex with respect to one *numéraire* can become concave with respect to another one. Finally, the *book of blueprints* they use is very limited in the sense that they work either with *wage-profit curves* of one country or at most with pairwise comparisons (as in the case of Han and Schefold (2006)). Furthermore, they do not compute the value of capital by using the production prices as it is done in this article (see eq. 1.13), instead they infer properties by studying the *wage-profit curves*, see also Schefold (2013a).

Given that we have the exact shape of the *wage profit frontier*, we have been in the position to compute the capital-labour ratio and to verify whether the neoclassical postulate that $dk/dr \leq 0$ holds for the case of *jelly capital* as well.

The correlation between the capital/output per employed ratio measured with the direct and indirect method (jelly capital) is very high. Therefore the qualitative results are very similar to those that were presented in the tables 4.2, 4.3 and the conclusions reached above still remain. For reason of space and because we believe the direct method to be more rigorous, we do not present this additional evidence, but it would suffice to say that they only confirm or strengthen the results. What we present here is evidence of *capital reversal* properties at *switch points* of the wage profit frontier. The bottom row of Table 4.5 is the number of *switch-points* per year[33].

The value of *jelly-capital* to the right of a *switch-point* is different from its value to the left of the same *switch-point*. We consider *capital reversal* to occur when the value of the *jelly capital* per labour to the right of the switch-point is higher than the one to the left[34].

Each row, except the last, of table 4.5 is the number of *switch-points* where *capital reversal* relative to the row's *numéraire* (η) and Net Product (Subsystem producing η). Note that the results change as the *numéraire* changes. We find that there is an overwhelming presence of *capital reversing*. The fact that both the direct method and the indirect method lead to the same results lends additional support to the robustness of the results presented here. Strong evidence in favour of the surrogate production function would have required a table filled with 1s for 4.2 and a table filled with 0s for table 4.5.

5 Conclusion: Aggregate production functions are *NOT* neoclassical

Samuelson (1962) attempted to set the foundations so that a system of heterogeneous production could be represented *as-if* it were a homogeneous production. Samuelson's *Surrogate Production Function* was challenged during the *Cambridge Capital Controversy* in the 60s. The special issue of the *Quarterly Journal of Economics*, also known as the '*QJE Symposium*', was devoted entirely to this debate. On that occasion, Samuel-

Table 4.5: ***Capital Reverse at Switch Points. Jelly Capital.*** *Number of Switch points where Capital Reverse occurs.*

Numéraire (η) and Net Product (Subsystem producing η)	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	1995-2011
1. Agriculture, Hunting, Forestry and Fishing	48	49	33	49	43	32	30	37	28	33	40	32	24	27	28	34	24	57
2. Mining and Quarrying	15	13	4	21	12	16	12	13	7	8	8	6	4	0	20	23	25	43
3. Food, Beverages and Tobacco	3	4	1	2	2	2	1	0	0	1	0	1	0	1	0	0	0	0
4. Textiles and Textile Products	3	0	0	4	3	0	1	0	0	0	3	0	0	4	10	14	15	14
5. Leather, Leather and Footwear	3	2	0	0	0	0	8	0	0	0	0	0	0	0	0	3	0	0
6. Wood and Products of Wood and Cork	1	1	2	6	9	6	8	4	0	4	8	2	8	8	3	4	2	10
7. Pulp, Paper, Paper , Printing and Publishing	0	2	2	5	0	4	0	4	0	2	7	3	1	7	11	12	4	13
8. Coke, Refined Petroleum and Nuclear Fuel	0	0	0	2	1	13	0	0	0	0	0	0	0	0	0	0	0	0
9. Chemicals and Chemical Products	0	0	0	0	0	0	0	0	0	9	16	7	5	6	10	12	16	20
10. Rubber and Plastics	0	0	0	0	0	0	0	4	1	3	1	3	0	4	0	0	0	2
11. Other Non-Metallic Mineral	13	9	4	9	6	10	5	5	3	9	16	9	12	5	1	3	6	18
12. Basic Metals and Fabricated Metal	9	12	9	1	5	1	2	0	0	0	4	3	7	0	13	7	11	23
13. Machinery, Nec	0	0	0	0	4	1	1	0	1	0	0	1	1	4	7	12	11	17
14. Electrical and Optical Equipment	0	0	0	4	0	0	0	0	0	1	0	0	0	0	6	15	12	8
15. Transport Equipment	15	16	7	17	6	10	5	11	0	1	1	0	0	5	6	15	12	6
16. Manufacturing, Nec; Recycling	18	19	12	14	2	7	7	4	3	11	18	3	0	3	0	0	0	0
17. Electricity, Gas and Water Supply	31	33	26	26	16	12	4	6	8	11	8	6	4	2	10	0	7	20
18. Construction	12	13	4	9	16	12	15	16	17	20	23	18	14	18	26	24	19	33
19. Sale, Maintenance and Repair of Motor Vehicles and Motorcycles; Retail Sale of Fuel	42	44	39	46	44	34	29	37	33	28	52	41	37	45	31	40	34	68
20. Wholesale Trade and Commission Trade, Except of Motor Vehicles and Motorcycles	54	56	38	50	44	34	30	37	33	40	55	43	26	45	51	46	41	76
21. Retail Trade, Except of Motor Vehicles and Motorcycles; Repair of Household Goods	60	60	44	57	60	46	41	44	41	38	58	46	39	50	51	53	45	78
22. Hotels and Restaurants	59	61	43	55	54	44	38	44	34	37	46	37	30	44	42	50	39	77
23. Inland Transport	42	37	20	34	30	28	24	31	16	19	23	24	7	28	26	27	20	60
24. Water Transport	19	17	6	19	10	2	1	0	2	1	0	0	1	5	6	2	1	4
25. Air Transport	9	5	1	12	4	2	1	18	10	6	0	0	0	1	1	0	1	7
26. Other Supporting and Auxiliary Transport Activities; Activities of Travel Agencies	32	28	16	35	23	18	12	25	14	27	50	33	30	40	39	46	37	62
27. Post and Telecommunications	46	41	14	26	31	24	23	38	20	27	39	36	24	41	40	40	37	61
28. Financial Intermediation	54	54	41	52	52	31	30	33	33	32	47	30	27	34	19	29	24	57
29. Real Estate Activities	2	0	0	3	0	1	6	11	1	12	19	9	6	10	11	7	4	8
30. Renting of M	49	43	32	47	39	26	23	28	28	23	36	31	20	34	30	38	32	52
31. Other Community, Social and Personal Services	47	54	33	47	40	30	26	33	31	34	52	42	36	47	43	47	39	65
32. Linear Combination of the Surplus of the 31 Subsystems	11	8	0	9	0	0	1	1	1	2	6	0	0	9	13	15	4	7
Total Number of Interval in the Wage-Profit Frontier	79	74	59	70	78	66	59	59	61	67	73	66	64	60	65	65	66	100

Each row is relative to the number of switch points for which there is *capital reversal*. The last row is the total number of switch points present on the *wage-profit frontier*.

son admitted that some problems do exist:

> Pathology illuminates healthy physiology. Pasinetti, Morishima, Bruno-Burnmeister-Sheshinski, Garegnani merit our gratitude for demonstrating that reswitching is a logical possibility in any technology, indecomposable or decomposable ... There often turns out to be no unambiguous way of characterizing different processes as more "capital-intensive," more "mechanized," more "roundabout" ... (Samuelson, 1966, p.582-3).

What was recognized as a pathology was the possibility that for some regions in the domain of the profit rate r, the change in the value of the capital-labour ratio could be positive, and not negative as required by *neoclassical theory*.

Like Samuelson, there were others who did admit ~~to~~ the existence of this problem, but it was considered to be a *pathology*, or a *perversity* or a *paradox*. Lacking empirical evidence, there has been a general tendency to declare a sort of faith regarding the tenability of the *neoclassical* cases (see on this Carter (2011)). At the end of the 1960s Ferguson wrote:

> [The] validity [of the Cambridge Criticism of neoclassical theory] is unquestionable, but its importance is an empirical or an econometric matter that depends upon the amount of substitutability there is in the system. Until *econometricians have the answer for us* placing reliance upon neoclassical economic theory is *a matter of faith. I personally have the faith* (Ferguson, 1969, p. xv; emphasis added).

Surprisingly, econometricians or economists have never really delivered a satisfactory answer. Macroeconomic theory has increasingly adopted the *Robinson Crusoe* type models, where capital is homogeneous with output and the Cobb-Douglas or CES type production functions are assumed to hold. Unfortunately, this has no empirical justification.

The highly questionable practice of assuming Cobb-Douglas like production functions *a priori* has not been abandoned even when studies have seriously disputed the statistical validity of the empirical estimations of the aggregated functions (Simon and Levy, 1963; Simon, 1979; Shaikh, 1974). What these authors have shown is that the seemingly robust estimation results are due to the *'laws of algebra'*, i.e., practically any data can be fitted with a Cobb-Douglas like production function. Hence, Cobb-Douglas and CES production functions cannot be taken as valid representations of production in an economy.

In this article, we have computed the methods belonging to the *wage-profit frontier* and we have computed all the possible *surrogate as-if* aggregated values following rigorous and robust methods as suggested in Samuelson (1962). For reasons that have been explained above, it is the first time that a rigorous computation of the *surrogate* production function and capital has been performed.

The evidence that we have provided leads us to conclude that the *surrogate* production function does not have the desired *neoclassical* properties. This conclusion, in light of the different criteria that have been used in the paper, is quite robust.

The very notion of aggregate marginal productivity, for the case of heterogeneous production, has no meaning. The consequences for economic theory should be obvious. The dominant macroeconomic *fairy tales* [35] are just *fairy tales* and they might have nothing or very little to do with the reality of the economic system As Samuelson said:

> If all of this causes headaches for those nostalgic for the old time parables of neoclassical writing, we must remind ourselves that scholars are not born to leave an easy existence. We must respect, and appraise, the facts of life. (Samuelson, 1966, p.582-3).

A Appendix. The *Neoclassical* Production Function - A Short Review

A.1 The Aggregate Neoclassical Production Function

Economists often assume that the production of an economic system can be described with an *as-if* production function. The neoclassical aggregate production function is a mathematical relation that links the output with the inputs and which holds specific properties (Solow, 1955, 1956, 1957; Arrow et al., 1961; Ferguson, 1969; Shephard, 1970). We consider the simple case with one output Y and two physical inputs, K, L.

$$Y = F(K, L) \tag{A.1}$$

There are three basic sets of assumptions that are often considered to be necessary for the above functional form to be *neoclassical*.
Set of Assumptions I: *Law of positive, but decreasing marginal productivities*
These assumptions are those of *convexity*, *continuity* and *differentiability*. This translates to the following properties for the function F: $\partial Y/\partial K = F_K(K, L) > 0$; $\partial^2 Y/\partial K^2 = F_{KK}(K, L) < 0$; $\partial Y/\partial L = F_L(K, L) > 0$, $\partial^2 Y/\partial L^2 = F_{LL}(K, L) < 0$. These, in turn, are characterized as the *law of positive, but decreasing marginal productivities*.
Set of Assumptions II: *Theory of Social Distribution based on Marginal Productivities*
The profit function is given by the following accounting relation:

$$\Pi = pY - wL - rK \tag{A.2}$$

The first order conditions for the maximization of profits are given by:

$$\frac{\partial \Pi}{\partial K} = p\frac{\partial Y}{\partial K} - r = 0 \Longrightarrow r = pF_K = p\frac{\partial Y}{\partial K} \tag{A.3}$$

$$\frac{\partial \Pi}{\partial L} = p\frac{\partial Y}{\partial L} - w = 0 \Longrightarrow w = pF_L = p\frac{\partial Y}{\partial L} \tag{A.4}$$

Equation A.3 is the *demand schedule for capital* and eq.A.4 is the *demand schedule for labour*. The physical world of production and that of the exchange maybe linked considering the Marginal Rate of Technical Substitution (MRTS), which is the change of one factor that is necessary to accommodate a change of another factor, so as to keep the production along the same isoquant. We have the following relation:

$$0 = dY = \frac{\partial Y}{\partial K}dK + \frac{\partial Y}{\partial L}dL \Rightarrow MRTS = -\frac{dK}{dL} = \frac{\partial Y}{\partial L}\Big/\frac{\partial Y}{\partial K} \tag{A.5}$$

Substituting eq.A.3 and eq. A.4 into A.5 we can link a technical relation with factor prices:

$$MRTS = -\frac{dK}{dL} = \frac{w}{r} \tag{A.6}$$

Set of Assumptions III: *Homogeneity of degree 1 and Constant Elasticity of Substitution - CES*
Arrow et al. (1961) introduce an additional feature, which include homogeneity of degree 1 - i.e. $F(\lambda K, \lambda L) = \lambda F(K, L) = \lambda Y$ - and Constant Elasticity of Substitution. The elasticity of substitution is given by:

$$\sigma = -\frac{dK/K}{dL/L}\Big/\frac{dMRTS}{MRTS} = \frac{\partial ln(K/L)}{\partial ln(MRTS)} = \frac{\partial ln(K/L)}{\partial ln(w/r)} \tag{A.7}$$

Clearly, with $\sigma = 1$, an increase in the capital-labour ratio will be matched by an exact increase in the *wage-profit ratio*. This is the case in which although it is possible to observe an increase in the capital-labour ratio, this will be associated with constant shares ($\frac{wL}{pY}$ and $\frac{rK}{pY}$).

There are different functional forms that would be consistent with the above set of assumptions[36]. The important point which is of interest concerns the relationship that is postulated between the factor prices r and w and the marginal productivities of capital and labour. The empirical verification of the tenability of the above *neoclassical* production function requires that $d(K/L)/dr < 0$ (or $\Delta(K/L)/\Delta r < 0$). If one finds evidence of the contrary the production function is NOT neoclassical.

A.2 The per-capita aggregate production function

The set of relations shown above can be rewritten for the case in which we consider the output-labour ratio $y = Y/L$ and the capital labour ratio $k = K/L$ as in Harcourt (1972, p.143), Arrow et al. (1961, p.229) or Ferguson (1969, p.253). In this case the properties required for the aggregate production functions to be neoclassical might become clearer. The production function may be rewritten, when homogeneous of degree 1, as:

$$y = \frac{Y}{L} = F(\frac{K}{L}, 1) = f(k) \tag{A.8}$$

with $f'(k) \geq 0$ and $f''(k) \leq 0$.

The firms' profit function is given by:

$$\pi = \frac{\Pi}{L} = p\frac{Y}{L} - rp\frac{K}{L} - w = pf(k) - rpk - w \tag{A.9}$$

The neoclassical relations or distributional assumptions (i.e., the first order conditions for the maximization of profits) require that (when $p = 1$):

$$r = f'(k) \geq 0 \tag{A.10}$$
$$\frac{dr}{dk} = f''(k) \leq 0 \tag{A.11}$$
$$\frac{dk}{dr} = \frac{1}{f''(k)} \leq 0 \tag{A.12}$$

and

$$w = f(k) - rk = f(k) - f'(k)k \geq 0 \tag{A.13}$$
$$\frac{dw}{dk} = f'(k) - f'(k) - f''(k)k = -f''(k)k \geq 0 \tag{A.14}$$
$$\frac{dk}{dw} = -\frac{1}{f''(k)k} \geq 0 \tag{A.15}$$

Clearly, an empirical verification of the existence o the *neoclassical* production function would require observations to confirm that $d(k)/dr < 0$ (or $\Delta(K/L)/\Delta r < 0$), see eq. A.12 or $d(k)/dw > 0$ (or $\Delta(K/L)/\Delta w > 0$), see A.15. If one finds evidence regarding the contrary, the production function is NOT neoclassical.

By combining relations A.11 and A.14 we obtain

$$\frac{dw}{dr} = \frac{dw}{dk} / \frac{dr}{dk} = \frac{-f''(k)k}{f''(k)} = -k \tag{A.16}$$

This is known in the literature as the *Jelly Capital*, Samuelson (1962, pp.200), Harcourt (1972, pp.143) and Felipe and McCombie (2013)

A.3 The multiple-factor multiple-firms case

The above can be extended to the multiple-factor, multiple-firms case. See for example Mas-Colell et al. (1995, Ch.5) for a fully neoclassical case, where stringent neoclassical assumption on the production function are postulated and Fisher (1992) and Felipe and McCombie (2013) for critical discussions.

We consider the case of a single production of commodity y_i having as inputs n capital goods and labour. The production function of commodity i may be rewritten, when homogeneous of degree 1, as:

$$y_i = \frac{Y_i}{L_i} = F\left(\frac{K_{i1}, K_{i2}, \ldots, K_{ij}, \ldots, K_n}{L_i}, 1\right) = f(k_{i1}, k_{i2}, \ldots, k_{ij}, \ldots, k_{in}) \tag{A.17}$$

with $f'(k_{ij}) \geq 0$ and $f''(k_{ij}) \leq 0$ for $j = 1, \ldots, n$.

The profit function is given by:

$$\pi_i = \frac{\Pi}{L_i} = \qquad p_i f_i(k_{i1}, k_{i2}, \ldots, k_{ij}, \ldots, k_{in}) - \tag{A.18}$$

$$- \quad r_1 p_1 k_{i1} - r_2 p_2 k_{i2} - \cdots - r_j p_j k_{ij} - \cdots - r_n p_n k_{in} - w \tag{A.19}$$

with $f'(k_{ij}) \geq 0$ and $f''(k_{ij}) \leq 0$ for $j = 1, \ldots, n$. First order conditions for the maximizations of profits (for fixed prices)[37]:

$$r_j = \frac{p_i}{p_j} f_i'(k_{ij}) \geq 0 \tag{A.20}$$

$$\frac{dr_j}{dk_{ij}} = \frac{p_i}{p_j} f_i''(k_{ij}) \leq 0 \tag{A.21}$$

$$\frac{dk_{ij}}{dr_i} = \frac{p_j}{p_i} \frac{1}{f_i''(k)} \leq 0 \tag{A.22}$$

Clearly as the *rental cost of capital* r_j increases the demand of the factor of production k_{ij} increases.

If we simplify by assuming uniform rate of profits, $r = r_1 = r_2 = \cdots = r_j = \cdots = r_n$

$$w = p_i f_i(k_{i1}, k_{i2}, \ldots, k_{ij}, \ldots, k_{in}) - r p_1 k_{i1} - r p_2 k_{i2} - \cdots - r p_j k_{ij} - \cdots - r p_n k_{in} \tag{A.23}$$

$$\frac{dw}{dk_{ij}} = p_i f'(k_{ij}) - r p_j - p_j k_{ij} \frac{dr}{dk_{ij}} \tag{A.24}$$

From eq. A.20 we know that $p_i f_i'(k_{ij}) - p_j r_j = 0$ and substituting eq. A.21 into the above we obtain:

$$\frac{dw}{dk_{ij}} = -p_j k_{ij} \frac{p_i}{p_j} f_i''(k_{ij}) = -p_i k_{ij} f_i''(k_{ij}) \geq 0 \qquad \forall k_{ij} > 0 \tag{A.25}$$

$$\frac{dk_i j}{dw} = -\frac{1}{p_i k_{ij} f''(k_{ij})} \geq 0 \qquad \forall k_{ij} > 0 \tag{A.26}$$

By combining relation A.21 and A.25 we obtain:

$$\frac{dw}{dr} = \frac{dw}{dk_{ij}} / \frac{dr}{dk_{ij}} = \frac{-p_i k_{ij} f''(k_{ij})}{p_i / p_j f_i''(k_{ij})} = -p_j k_{ij} \tag{A.27}$$

We can also extend this to the case in which the change in the demand of the means of production has an impact on the prices.

Notes

[1]See Appendix for a definition of the aggregate neoclassical production function. See also Arrow et al. (1961), Ferguson (1969), Zellner and Revankar (1969)

[2] For example, modern growth theory is based on the various developments of the *well-behaved* Solow-Swan or Ramsey-Cass-Koopmans models. Endogenous growth models, Overlapping Generations Model, Real Business Cycles, aggregate demand and aggregate supply, Dynamic Stochastic General Equilibrium, Computable General Equilibrium models and so on presuppose neoclassical production functions. Due to limited space, we do not review the various strands of development of this literature. However, the interested reader may refer to: **?**, Romer (2011), Galí (2008), Gong and Semmler (2006), McCandless (2008).

[3]For example, consider two firms, producing output with the following production functions $Y_1 = F(K_1, L_1)$ and $Y_2 = G(K_2, L_2)$, respectively. Even if they both satisfy the usual neoclassical properties, i.e., positive marginal productivities and positive marginal rates of substitution, it is possible that the function $F + G$ may not satisfy the above mentioned neoclassical properties. There is some recent econometric work which addresses the issue of aggregation, but they do not discuss the problems of the mapping between heterogeneous factors of production, i.e. capital, and heterogeneous output into the scalar index numbers K and Y. For instance, Dupuy (2012) has heterogeneous labour, but assumes that the workers are given one machine (hence, by definition, heterogeneity of the inputs other than labour is ruled out). See Ackerberg et al. (2007) for another study as an example, where structural parameters are estimated. However, the structural form of the generalized neoclassical production function is assumed to hold *a priori*.

[4]A recent review is to be found in Cohen and Harcourt (2003) and Pasinetti (2003), Felipe and McCombie (2003), Fisher (2003). A relevant list of contributions to the two Cambridges debate can also be found in Zambelli (2004)

[5]It is quite surprising that an empirical verification of this problem has never been attempted, atleast to the best of our knowledge. In a recent contribution on the identification of the elasticity of substitution of the aggregate production function, the authors acknowledged the existence of this problem. They wrote (León-Ledesma et al., 2010, p.1331, fn. 3) " *...the Cambridge capital controversy of the 1960s questioned the existence of aggregate production functions and thus the possibility of their econometric identification*". Instead, they decided " *to leave the issues raised by the Cambridge debate outside of [their] study*" and assumed a neoclassical production function from the outset.

[6] In this paper by *profits* it is meant the returns in value of the used means of production. In this sense the term *intrest rates* and *profit rates* should be considered as being equivalent

[7]We will proceed with the assumption that the different entries of the input-output tables are measured in *"physical"* units and not in *"value units"*. This is a very strong assumption. The input-output tables that we will use for the empirical verification have all been deflated by industry level price indexes. The purpose of the present paper is to check whether there is a relation between the observations and the neoclassical production function. The neoclassical production function is formed by units of capital that are assumed *as-if* they could be expressed in terms of *physical units*. A reader who feels uneasy in considering the entries of the input-output tables as if they were physical, measurable units should also consistently reject the neoclassical production theory, where the aggregate capital is assumed to have a physical dimension, but it is computed as an average or sum of input-output tables entries. Hence, any critique on the use of input-output tables based on the distinction between values and quantities has to be extended to the aggregate as well. For an excellent survey on the problem of interpreting the input-output tables as if they were physical quantities, see Felipe and McCombie (2013, pp.41-47)

[8]By replacing a with k, ℓ with 1 and b with y we have the inputs necessary for eq. A.17. We have decided to keep the observations in the input-output data and the inputs and outputs present in the *fairy tale* neoclassical production function separate, but we could mix the two notations as well.

[9]

$$\mathbf{A^z} = \begin{bmatrix} a_{11}^{z_1} & a_{12}^{z_1} & \dots & a_{1n}^{z_1} \\ a_{21}^{z_2} & a_{22}^{z_2} & \dots & a_{2n}^{z_2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}^{z_n} & a_{n2}^{z_n} & \dots & a_{nn}^{z_n} \end{bmatrix};$$

$$\mathbf{L^z} = \begin{bmatrix} \ell_1^{z_1} \\ \ell_2^{z_2} \\ \vdots \\ \ell_n^{z_n} \end{bmatrix}; \mathbf{B^z} = \begin{bmatrix} b_1^{z_1} & 0 & \dots & 0 \\ 0 & b_2^{z_2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & b_n^{z_n} \end{bmatrix};$$

[10]Koopmans has the following definition of 'activity':

'[A]n activity consists of the combination of certain qualitatively defined commodities in fixed quantitative ratios as "inputs" to produce as "outputs" certain other commodities in fixed quantitative ratios to the inputs' Koopmans (1951, pp.35-6).

[11] We assume that the reader is familiar with this notation. See for example Sraffa (1960) or Pasinetti (1977).

[12] Equation 1.3 is consistent with the case in which the activity level is the unit vector. If we consider a different level of activity, $\mathbf{x}$ and define the diagonal matrix $\mathbf{X} = diag(\mathbf{x})$ the following equation 1.2 becomes $\mathbf{XA^z}(1+r)\mathbf{p} + \mathbf{XL^z}w = \mathbf{XB^z p}$. Therefore the price vector is given by $\mathbf{p^z}(r,w) = [\mathbf{X}(\mathbf{B^z} - \mathbf{A^z}(1+r))]^{-1}\mathbf{XL^z}w$. Please note that a straight forward algebraic manipulation leads to $\mathbf{p^z}(r,w) = [\mathbf{B^z} - \mathbf{A^z}(1+r)]^{-1}\mathbf{X}^{-1}\mathbf{XL^z}w = [\mathbf{B^z} - \mathbf{A^z}(1+r)]^{-1}\mathbf{L^z}w$, where the term $\mathbf{X}$ disappears from the equation. Hence the prices are not dependent on the level of activity. This is the meaning of the *non-substitution theorem*. The expression *non-substitution theorem* might be misleading (see Pasinetti (1977, p.168)): the term *substitution* should not to be mistaken with the Marginal Rate of Technical Substitution (MRTS) defined in eq. A.5. In the sequel when the activity vector $\mathbf{x}$ is not explicitly included in the equations, it is because the equation *"holds for any feasible level of activity"* $\mathbf{x}$.

[13]The number of possible curves, s, is enormous. In our database, there are 31 sectors and 30 countries. In order to determine the yearly *wage-profit frontier*, we would need to compute, in the absence of an efficient method, $31^{30}(\approx 5.5 \times 10^{44})$ *wage-profit curves*, relative to one year, or $31^{30\times 17}(\approx 3.6 \times 10^{760})$ curves when we consider the whole time period (17 years). Either one can computes all these curves first one by one, which is impossible (because of time and memory space constraints), or one should use an algorithm that reduces the computational time. Fortunately, this algorithm has been developed and explained in Zambelli et al. (2014).

[14] An excellent descriptionof the wage-profit frontier (also called technological frontier of factor price frontier) is to be found in (Pasinetti, 1977, Ch.6). See also Felipe and McCombie (2013, p.36 and the subsequent) or Harcourt (1972, Ch.4)

[15]Schefold (2013b) argues that the number of wage profit curves at the frontier has to be low. The results of this paper provide supporting evidence (see below table 4.5, bottom row).

[16] It is assumed here that $\mathbf{Y^{\tilde{z}}}$ is semipositive, i.e the entries of the surplus are zero or positive.

[17] Some may question our construction of the surrogate production function on the basis that the neoclassical production functions have fixed capital and labour as inputs, while here we work exclusively with circulating capital, i.e. the means of production used during the year. Our measurement of capital is based on eq. 1.13 $K_{val}^{\mathbf{z}}(r,\mathbf{x},\eta) = \mathbf{x}'\mathbf{A^z}\mathbf{p^z}(r,\eta)$. This might not be entirely satisfactory, but it is difficult to figure out alternative ways to measure capital. In the case of the aggregate neoclassical production function, capital is homogeneous with respect to output. Estimates of fixed capital are usually based on the *perpetual inventory methods*, which uses the yearly flow values of the input-output tables. Although attempts to find robust indexes (or proxies) to measure capital (as if it was a physical magnitude) are made, the point of departure is at the transformations that directly use the input-output tables observations. A fraction of the value of output produced is assumed to be used for the fixed capital formation (investment), while a fraction δ is assumed to be the depreciation of capital. The question is whether the use of the factors of production should be negatively related to factor prices. This should occur for the circulating capital as well, which is a highly correlated component with respect to the total value of capital. In any case, (Samuelson, 1962, p.201) *as-if* notion of *surrogate capital* is, in our view, conceptually the same as the one proposed here with eq. 1.13. The case with *fixed capital* is computed below (section

4.4) with the indirect method.

[18]In the literature we find *Capital Reversing* and *Reverse Capital Deepening*. *Capital reversing*, known also as *real (or price) Wicksell effect* occurs when the physical (or value of) capital increases, while the profit rate and the quantity of labour used increases. Neoclassical production function would imply that $dk/dr < 0$. *Reswitching* (or *reverse capital deepening*) occurs when at least one *wage-profit curves* belonging to the *wage-profit frontier* have two or more switch points. This implies the existence of *capital reversing*. It is a special case that has received a lot of attention during the two Cambridges debate, but it has very little empirical importance.

In order to assess whether or not a system of methods is consistent with neoclassical postulates, it is sufficient to search for *capital reversing* cases. In fact, reswitching or *reverse capital deepening* both imply *capital reversing*, which is a sufficient condition for the exclusion of a combination of methods as being neoclassical. Some authors seem to consider both concepts as being synonymous. For example Harcourt (1972, p.8) defines *'Capital-reversing'* as *the possibility of a positive relationship between the value of capital and the rate of profits.*. Similarly Kurz and Salvadori (1998, p.416) define *'reverse capital deepening'* as *. . . the possibility that in a multisectoral economy the relationship between capital per unit of labour and rate of profit . . . may be increasing*. On the contrary others seem to make a distinction between the two notions by linking *reverse capital deepening* with *'double-switching'*, that is when the efficient method of production for one commodity occurs only in disjoint intervals of the domain of the wage profit frontier.

> The phenomenon that has attracted most attention is that of reswitching and reverse capital deepening: there may be switch points on the original envelope such that the intensity of capital does not fall with the rate of profit (reverse capital deepening) **and** the individual wage curve may have appeared on the envelope already at a lower rate of profit (reswitching). (Schefold, 2013a, p.1164, emphasis added).

In this paper, we will consider *Capital Reverse* and *Reverse Capital Deepening* as the cases in which a negation of eq. 1.18 occurs.

[19] For a more detailed explanation see also Pasinetti (1977, Ch. 6) or Felipe and McCombie (2013, p.36 and the subsequent).

[20]Once the parameters have been estimated so as to provide the best fit to a given neoclassical production function, it is not surprising that the estimated function would be a best fit and the derived Labour Demand and Capital Demand curves would be *well-behaved*. This would be so by construction. Fig. 1.1 is an example. Once a Cobb-Douglas function is postulated, the figures would be *well-behaved* by definition. For obvious reasons, the existing econometric work based on this presupposition cannot be used as evidence for the tenability of the neoclassical production function.

[21] The value of $\mathbf{x}^*_{r,\eta}$ is obtained by solving the following linear programming problem , where the total cost associated to the fixed value of net output is minimized. The total cost is given by the sum of the cost of the employed means of production and the cost of employed labour:

$$\text{Total Cost} = \mathcal{K}^{\tilde{\mathbf{z}}}_{val}(r, \mathbf{x}, \eta) + \mathcal{L}^{\tilde{\mathbf{z}}}(\mathbf{x})w^{\tilde{\mathbf{z}}}(r, \eta) = \mathbf{x}'\mathbf{A}^{\tilde{\mathbf{z}}}\mathbf{p}^{\tilde{\mathbf{z}}}(r, \eta) + \mathbf{x}'\mathbf{L}^{\tilde{\mathbf{z}}}w^{\tilde{\mathbf{z}}}(r, \eta) \tag{A.28}$$

The cost minimization is given by solving the following linear programming problem:

$$\min_{\mathbf{x}} \left[\mathbf{x}'\mathbf{A}^{\tilde{\mathbf{z}}}\mathbf{p}^{\tilde{\mathbf{z}}}(r, \eta) + \mathbf{x}'\mathbf{L}^{\tilde{\mathbf{z}}}w^{\tilde{\mathbf{z}}}(r, \eta)\right] \tag{A.29}$$

$$\text{s.t. } \mathbf{x}'\left[(\mathbf{B}^{\tilde{\mathbf{z}}} - \mathbf{A}^{\tilde{\mathbf{z}}})\right] \geq \mathbf{0}' \tag{A.30}$$

$$\mathbf{x}'\left[(\mathbf{B}^{\tilde{\mathbf{z}}} - \mathbf{A}^{\tilde{\mathbf{z}}})\right]\mathbf{p}^{\tilde{\mathbf{z}}}(r, \eta) = \overline{\mathcal{Y}}^{\tilde{\mathbf{z}}}_{val} \tag{A.31}$$

$$\mathbf{x}' \geq \mathbf{0}' \tag{A.32}$$

As explained above in section 1.3 there are v different intervals forming the *wage-profit frontier*. We simplify notation by identifying the set of methods belonging to each interval with $\tilde{\mathbf{z}}$. Hence, once r is given, also the efficient set of methods is given or identified $\tilde{\mathbf{z}}$. A change in the activity vector $\mathbf{x}$, due to the *non-substitution theorem* will not change the prices, $\mathbf{p}^{\tilde{\mathbf{z}}}(r, \eta)$, which would depend exclusively on r and on η. In this paper we aim at constructing the aggregated isoquants. In the case of the isoproduct, once the surplus vector is fixed and the profit rate is also given, the computation of the activity vector is straightforward as shown by eq.2.1.

Here, for the case of the isovalue, we have that the value of the net output is kept fixed (this is captured by eq.A.31). Clearly, differently from the case of the isoproduct, the surplus vector (eq. A.30) would change as the activity vector changes. The use of linear programming has only a technical meaning, i.e. it is a tool useful for the construction of the so-called efficient isovalue or isoquant curves. Nothing is implied in terms of individual agent's choices or actual market behaviour.

[22]The countries considered are: (AUS) Australia; (FIN) Finland; (KOR) Korea;(AUT) Austria; (FRA) France; (MEX) Mexico; (BEL) Belgium; (GBR) Great Britain; (NLD) Netherlands; (BRA) Brazil; (GRC) Greece; (POL) Poland; (CAN) Canada; (HUN) Hungary; (PRT) Portugal; (CHN) China; (IDN) Indonesia; (RUS) Russia; (CZE) Czech Republic; (IND) India; (SWE) Sweden; (DEU) Germany; (IRL) Ireland; (TUR) Turkey; (DNK) Denmark; (ITA) Italy; (TWN) Taiwan; (ESP) Spain; (JPN) Japan; (USA) United States.

[23]The excluded sectors are: Public Administration and Defence, Compulsory Social Security, Education, Health and Social Work, Private Households with Employed Persons. These sectors use substantial amount of inputs for the remaining sectors, but provide very little inputs with respect to the other sectors. In a way they can be seen as non-basic commodities (Sraffa, 1960, pp.6-7).

[24]What described are sectorial or industry subsystems as in Sraffa (1960, Appendix A, p.89), Gossling and Dovring (1966), Gossling (1972).

[25]The use of the term constructive is to be understood in the same way as Velupillai (1989, 2008) describes Sraffa's method as being constructive. We take only the observed "quantities" of the input-output tables as given and provide constructive or effective procedure to determine or compute the necessary magnitudes.

[26]That is, 32 different *numéraires* (31 composed of one commodity each and a numéraire which is a bundle made of all the industries) for 18 time intervals (17 years and the whole period from 1995-2011). In the case of the computations of the Isoproducts, the fixed Net Surplus was set to be equal to the *numéraire* as well. The number of intervals per year are reported in the last row of Table 4.5. Each entry of all the tables below are relative to the case in which the numéraire is the social surplus of the commodity indicagted in the row (for example row 1 would be *Agriculture, Hunting, Forestry and Fishing*, row two *Mining and Quarrying* and so on).

[27]This implies the following relation:

$$\overline{\mathbf{y}} \quad = \quad \overline{\mathbf{x}}_r'(\mathbf{B}^{\tilde{\mathbf{z}}} - \mathbf{A}^{\tilde{\mathbf{z}}}) \qquad \tilde{\mathbf{z}} \in \mathbf{Z}_{\boldsymbol{E}}^{\mathrm{WPF}} \tag{A.33}$$

[28]Only one method changes at switch points, see Bharadwaj (1970); Pasinetti (1977); Zambelli et al. (2014)

[29]We considered 576 instances (32 *numéraires times* 17 years) and for each instances we looked at the behaviour of the 31 industries, for a total of 17856 (576 *times* 31) individual industry level capital-output ratios per worker functions

[30]The direct method to compute the aggregate value of output might be criticized because we are working with circulating capital. In footnote 17 we have made a case for the computation of the neoclassical or Samuelson's *surrogate* capital with eq. eq. 1.13, saying that the circulating capital is indeed correlated with the capital formation values. Here *fairy tale* Capital has to be interpreted as fixed and circulating capital. Hence, the potential critique for this verification does not hold.

[31]On the definition of *capital reversing* or *reswitching* see above footnote 18.

[32]The *sufficient* condition is when all the individual industry level production functions have all the neoclassical shapes, see below appendix A.3, eq. A.27. The case reported in Figure 4.4 above is a good example of a seemingly neoclassical aggregate production function. Clearly the *wage-profit curve* (North-West) is convex to the origin. This means that the computed aggregate *capital-labour ratio per employed* is negatively sloped with respect to the profit rate and this is indeed the case. The qualitative behaviour of the capital-output ratio per capita would be qualitatively similar to the qualitative behaviour that we have computed with the direct method (top graph of Fig. 4.4). But as we see from figure 4.5, the industry level production functions are not at all neoclassical.

[33]The few *wage-profit curves* or *switch-points* at the frontier confirm Schefold (2013b) conjecture that in

spite of a very large number of *wage-profit curves*, the relevant ones are only a few.

[34]To be precise, dw/dr is almost always not differentiable at the switch points. Therefore, we should compute the left and right limits of the *switch-point* using numerical approximation. To exclude *capital reversing*, the sufficient condition is that the approximating points in the interval have to be convex with respect to the origin.

[35] *Fairy tale* is how Samuelson defined in 1962 his *surrogate production function*, see the quotation reported above at page 5.

[36]Arrow et al. (1961) have suggested the following generalized CES-Production function $Y = F(K, L) = \gamma_1[K^\rho + \gamma_2 L^\rho]^{1/\rho}$ where $\rho = (\sigma - 1)/\sigma$.

[37]Clearly, when we consider interdependencies the assumption of fixed prices is a rather strong one. In the literature the case where a change in the demand of the means of production does affect the prices is seldom considered. For example, see Mas-Colell et al. (1995, pp.146-152), where prices are assumed to be constant. Here, we consider the fixed-price condition only for the sake of the exposition. As we will see below, accounting equilibrium would require that as the wage and the profit rates change, the prices must change as well.

References

Ackerberg, D., L. Benkard, S. Berry, and A. Pakes (2007). Econometric tools for analyzing market outcomes. In J. Heckman and E. Leamer (Eds.), *Handbook of Econometrics*, Volume 6A. North Holland.

Arrow, K. (1951). Alternative proof of the substitution theorem for Leontief models in the general case. In T. Koopmans (Ed.), *Activity Analysis of Production and Allocation*, pp. 155–164. New York: John Wiley.

Arrow, K., S. Chenery, B. Minhas, and R. M. Solow (1961). Capital-labor substitution and economic efficiency. *The Review of Economics and Statistics 43*(3), pp.225–250.

Bharadwaj, K. (1970). On the maximum number of switches between two production systems. *Schweizerische Zeitschrift für Volkswirtschaft und Statistik 106*(4), 409–429.

Bruno, M., E. Burmeister, and E. Sheshinski (1966). The nature and implications of the reswitching of techniques. *Quarterly Journal of Economics 80*(4), 526–553.

Carter, S. (2011). C.E. Ferguson and neoclassical theory of capital: A matter of faith. *Review of Political Economy 23*(3), 339–356.

Cekota, J. (1988). Technological change in canada (1961-80): An application of the surrogate wage function. *The Canadian Journal of Economics / Revue canadienne d'Economique 21*(2), pp. 348–358.

Cohen, A. J. and G. C. Harcourt (2003). Whatever happened to the Cambridge capital theory controversies? *Journal of Economic Perspectives 17*, 199–214.

Dietzenbacher, E., B. Los, R. Stehrer, M. Timmer, and G. de Vries (2013). The Construction of World Input- Output Tables in the WIOD Project. *Economic Systems Research 25, no. 1*, 71–98.

Dupuy, A. (2012, A microfoundation for production functions: Assignment of heterogeneous workers to heterogeneous jobs. *Economica 79*(315), 534–556.

Farrell, M. (1957). The measurement of productive efficiency. *Journal of the Royal Statistical Society. Series A (General). 120*(3), 253–290.

Felipe, J. and McCombie (2003). Comments: Cambridge capital controversies. *Journal of Economic P 17*, 229–30.

Felipe, J. and J. McCombie (2013). *The aggregate production function and the measurement of technical change. "Not Even Wrong".* Edward Elgar Publishing.

Ferguson, C. (1969). *The Neoclassical Theory of Production & Distribution*. Cambridge University Press, reprinted in 1979,.

Fisher, F. (1969). The existence of the aggregate production functions. *Econometrica 37*(4), 553–77.

Fisher, F. (2003). Comments: Cambridge capital controversies. *The Journal of Economic Perspectives 17*(4), 228–29.

Fisher, F. M. (1992). *Aggregation. Aggregate Production Functions and Related Topics: Collected Papers of Franklin M. Fisher*. MIT Press.

Fisher, I. (1922). *The Making of Index Numbers: A Study of Their Varieties, Tests, and Reliability*. Cambridge, MA.: The Riverside Press.

Frisch, R. (1936). Annual Survey of General Economic Theory: The Problem of Index Numbers. *Econometrica 4*(1), 1–38.

Galí, J. (2008). *Monetary Policy, Inflation, and the Busines Cycles*. New Jersey: Princeton University Press.

Garegnani, P. (1966). Switching of techniques. *Quarterly Journal of Economics 80*(4), 554–567.

Gong, G. and W. Semmler (2006). *Stochastic Dynamic Macroeconomics: Theory and Empirical Evidence*. Oxford University Press, New York, Inc.

Gossling, W. F. (1972). *Productivity Trends in a Sectoral Macro-Economic Model*. Input-Output Publishing Co., distributed by Frank Cass & Co., London.

Gossling, W. F. and F. Dovring (1966, May). Labor productivity measurement: The use of subsystems in the interindustry approach, and some approximating alternatives. *Journal of Farm Economics 48*(2), 369–377.

Han, Z. and B. Schefold (2006). An empirical investigation of paradoxes: reswitching and reverse capital deepening in capital theory. *Cambridge Journal of Economics 30*, 737–765.

Harcourt, G. (1972). *Some Cambridge controversies in the theory of capital*. Cambridge University Press.

Koopmans, T. (1951). Alternative proof of the substitution theorem for leontief models in the case of three industries. In T. Koopmans (Ed.), *Activity Analysis of Production and Allocation*, pp. 33–97. New York: John Wiley & Sons.

Krelle, W. (1977). Basic facts in capital theory. some lessons from the controversy in capital theory. *Rvue d'Economie Politique 87*, 282–329.

Kurz, H. and N. Salvadori (1998). Reverse capital deepening and the numeraire: A note. *Review of Political Economy 10*(4), 415–426.

León-Ledesma, M., P. McAdam, and A. Willman (2010). Identifying the elasticity of substitution with biased technical change. *American Economic Review 100*(4), 1330–1357.

Leontief, W. (1985). *Input-Output Economics*, Chapter Technological Change, Prices, Wages, and Rates of Return on Capital in the U.S. Economy, Chapter 19, pp. 392–417. New York: Oxford University Press.

Mariolis, T. and L. Tsoulfidis (2011). Eigenvalue distribution and the production price-profit rate relationship: theory and empirical evidence, evolutionary and institutional. *Evolutionary and Institutional Economics Review 8*(1), pp. 87–122.

Mas-Colell, A., M. D. Whinston, and J. Green (1995). *Microeconomic Theory*. New York: Oxford University Press.

McCandless, G. (2008). *The ABCs of RBCs: An Introduction to Dynamic Macroeconomic Models*. Harvard University Press, Cambridge, Massachusetts.

Ochoa, E. M. (1989). Values, prices and wage profit curves in the US economy. *Cambridge Journal of Economics 13*, 413–29.

Ozol, C. (1984). Parable and realism in production theory: The surrogate wage function. *The Canadian Journal of Economics / Revue canadienne d'Economique 17*(2), pp. 353–368.

Pasinetti, L. (1966). Changes in the rate of profits and switches of techniques. *Quarterly Journal of Economics 80*(4), pp. 503–517.

Pasinetti, L. (1977). *Lectures on the Theory of Production*. New York: Columbia University Press.

Pasinetti, L. (2003). Comments: Cambridge capital controversies. *Journal of Economic Perspectives 17*(4), 227–28.

Romer, D. (2011). *Advanced Macroeconomics* (4 ed.). New York: McGraw-Hill Irwin.

Samuelson, P. A. (1951). Abstract of a Theorem Concerning Substitutability in Open Leontief Models. In T. Koopmans (Ed.), *Activity Analysis of Production and Allocation*. New York: John Wiley.

Samuelson, P. A. (1962). Parable and realism in capital theory: the surrogate production function. *Review of Economic Studies 29*(3), 193–206.

Samuelson, P. A. (1966). A summing up. *Quarterly Journal of Economics 88*(4), 568–583.

Sato, K. (1974). The neoclassical postulate and the technology frontier in capital theory. *Quarterly Journal of Economics 88*(3), 353–384.

Schefold, B. (2013a). Approximate production functions. *Cambridge Journal of Economics 37*, 1161–1184.

Schefold, B. (2013b). Only a few techniques matter! on the number of curves on the wage frontier. In E. S. Levrero, P. A., and S. A. (Eds.), *Sraffa and the Reconstruction of Economic Theory: Volume 1. Theories of Value and Distribution.*, pp. 46–69. Palgrave.

Shaikh, A. (1974). Laws of production and laws of algebra: The humbug production function. *The Review of Economics and Statistics 56*(1), 115–120.

Shaikh, A. (1988). The empirical strength of the labor theory of value. In R. Bellofiore (Ed.), *Conference Proceedings of Marxian Economics: A Centenary Appraisal*. Macmillan, London.

Shaikh, A. (2012). The empirical linearity of Sraffa's critical output-capital ratios. In C. Gerke, N. Salvadori, I. Steedman, and R. Sturn (Eds.), *Classical Political Economy and Modern Theory. Essays in Honour of Heinz Kurz.* Routledge.

Shephard, R. (1970). *Theory of Cost and Production Functions*. New Jersey: Princeton University Press.

Simon, A. H. (1979). On parsimonious explanations of production relations. *The Scandinavian Journal of Economics 81*(4), 459–474.

Simon, A. H. and F. K. Levy (1963). A note on the Cobb-Douglas function. *Review of Economic Studies 30*(2), 93–94.

Solow, R. M. (1955). The production function and the theory of capital. *The Review of Economic Studies 23*(2), 101–108.

Solow, R. M. (1956). A contribution to the theory of economic growth. *Quarterly Journal of Economics 70*(1), 65–94.

Solow, R. M. (1957). Technical change and the aggregate production function. *The Review of Economics and Statistics 39*(3), 312–320.

Solow, R. M. (1976). Discussion of Zarembra's chapter - characterization of a technology in capital theory. In M. Brown, K. Sato, and P. Zarembra (Eds.), *Essays in Modern Capital Theory*, pp. 138–140. North-Holland.

Sraffa, P. (1960). *Production of Commodities by Means of Commodities*. Cambridge: Cambridge University Press.

Timmer, M. (Ed.) (2012). *The World Input-Output Database (WIOD): Contents, Sources and Methods, WIOD Working Paper Number 10, April 2012, downloadable at http://www.wiod.org/publications/papers/wiod10.pdf*. EU - 7th Framework Programme.

Velupillai, K, V. (1989). The existence of the standard system: Sraffa's constructive proof. *Political Economy Studies in the Surplus Approach 5*(1), 3–12.

Velupillai, K, V. (2008). Sraffas economics in non-classical mathematical modes. In G. Chiodi and L. Ditta (Eds.), *Sraffa or Alternative Economics*, Chapter 14, pp. 275–294. Palgrave Macmillan.

Zambelli, S. (2004). The 40% neoclassical aggregate theory of production. *Cambridge Journal of Economics 28*(1), 99–120.

Zambelli, S., T. Fredholm, and R. Venkatachalam (2014). Robust measurment of national technological progress. *ASSRU Discussion Paper, No. 4-2014, Algorithmic Social Sciences Research Unit, University of Trento*.

Zellner, A. and N. S. Revankar (1969, April). Generalized production functions. *Review of Economic Studies 36*(2), 241–250.