Algorithmic Social Sciences Research Unit

Remembering Krishna Bharadwaj✣

K. Vela Velupillai

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"In the work of Krishna Bharadwaj (1935-1992) who pursued Sraffa's trail in economic theory, the Indian student can lay claim to a double bonus. For she not only contributed famously to the Sraffian project, in its critical as well as *reconstructive aspects*, but also brought the 'open' structure of classical theory to bear on the problem of economic backwardness. … ."
Omkarnath, 2005, p. 459; italics added.

✣ Krishna Bharadwaj, whose Sraffian Scholarship was supreme, died on 8 March, *twenty years ago*. She began her remarkable journey towards a mastery of Sraffa (1960) in 1962, a half a century ago. My own studies of Sraffa began ten years later, in 1972, but it was only in 1992, the year Krishna died, that I was able to develop an applicable index, based on Sraffa's *magnum opus*. This particular saga was completed by Stefano Zambelli, *to whom I dedicate this paper*, who constructed an algorithm to use my index, based on a deep and abstract result in Bharadwaj (1970). I am indebted to my colleague and friend (of thirty years), Stefano Zambelli, whose own adherence to Sraffian methods has been a beacon of light in my eternal struggles with *Production of Commodities by Means of Commodities*.

Abstract

Om this, the 20th anniversary of the untimely death of Krishna Bharadwaj, I try to reconsider some theoretical aspects of her fundamental contributions to capital theory by showing how applicably relevant they are, in modern contexts. Krishna Bharadwaj had an admirable mastery of Sraffian methodology and remained loyal to that tradition in a most enlightened manner. Her theoretical contributions to Sraffian scholarship enhanced and enlarged the frontiers of applicable capital theory. One particular application of two *Bharadwaj Theorems* is also considered in this paper.

§1. A Preamble on Anniversaries, Realism & Parables

> "[Krishna Bharadwaj] 'did not then [in 1962] know of the more fundamental critique of economic theory heralded by Piero Sraffa's work' (Bharadwaj, 1992, p. 38). This came in 1962 when 'Sachin Chowdhury, the editor of *The Economic Weekly* [which subsequently became *The Economic and Political Weekly*], …. drew out of his drawer [Sraffa's] slim volume … [which Krishna …] agreed to review … in a month or so!' (ibid, p. 39)"
> Harcourt, 1993-94, p. 301; italics in the original.

Fifty years ago, just as Krishna Bharadwaj was being made aware of Sraffa's book, Peter Newman's significant review of Sraffa (1960, henceforth referred to as *PCMC*; Newman, 1962[1]) appeared, and elicited one of only two public reactions by Piero Sraffa to his slim, terse, book (the other was his response to Harrod's misleading review of *PCMC*, Harrod (1961)[2] and Sraffa (1962)). Ironically, as it seems now, 1962 was also the year Paul Samuelson's story of '*parables and realism in capital theory*' appeared (Samuelson, 1962), which has been given a new lease of life in Anwar Shaikh's recent reflections on the empirical (in)significance of reswitching (Shaikh, 2012).

Omkarnath (2005) has documented quite handsomely[3] the approval with which even the legendary silences of Sraffa – especially with regard to comments on his own contributions – were broken by the elegance and competence of Krishna Bharadwaj's maiden foray into the world of *PCMC*. I shall not traverse these well-trodden noble paths anew. I aim to use the traditional intellectual excuse of anniversaries to try to point out a neglected aspect of Sraffa's concerns on anchoring serious theory on observational realism, without idle construction of non-relevant parables that do not aid understanding reality but obfuscate and obscure the distinction between relevant abstractions and irrelevant approximations to a non-observable reality.

For this purpose I invoke two Bharadwaj Theorems, in the next section, which are then used as a spring board to compute an index number in a computationally efficient manner to compare two productions systems, characterizing two alternative economies. Neither Sraffa, nor Krishna Bharadwaj, indulged in 'abstracton-mongering' in 'ahistorical' intellectual exercises, without anchoring their theories in relevant realism, free of parables.

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[1] Professor Guglielmo Chiodi drew my attention to a hand-written letter by Sraffa to Gareganani, dated 22-6-62, referring to Newman's review as "È tipico effetto della troppa (e troppo poco digeribili) matematica", which in my 'free translation' would read: '[Newman's review is] a typical outcome of too much (and little digestible) mathematics.'

[2] Sraffa, in the above letter to Garegnani, refers to this review in blunt terms as: 'Harrod è una perdita di tempo' (again, a 'free translation' by me would render it as: 'Harrod is a waste of time.').

[3] Curiously, the one issue of **EPW** inaccessible via *JSTOR* is Vol. 27, # 12 (1992), which is supposed to contain Deena Khatkhate's 'Letter to the Editor' referring to Sraffa's own letter to Sachin Chowdhury stating that the Bharadwaj review of *PCMC* 'was one of the three best reviews' of his book (Omkarnath, op.cit, p. 460). Professor Omkarnath has kindly made available to me a copy of this *Letter to the Editor* of EPW, a perusal of which leaves the 'mystery' of the other two 'best reviews' unresolved, however.

Even in the context of the deepest issues of the theory of value, Sraffa's anchoring – and Krishna Bharadwaj's, too – was in the *observable* entities of production, and not pseudo-metaphysical psychological bases of so-called rational behavior. In a note written in summer 1928, as Heinz Kurz has recently reminded us in Kurz, 2011), Sraffa emphasizes (italics added):

> "The question asked of the theory of value is the following: *Given* (*from experience*) *the prices of all commodities ..., find a set of conditions that will make these prices appear to be necessary*. This means, *given the unknowns, find the equations* (i.e. the constants) … "
> (Sraffa Papers D3/12/9: 65)

The immediate implication is that the economic analyst faces a *Diophantine Decision Problem* (as I have argued in Velupillai, 2005). The 'given (from experience) prices' cannot be other than – at best – rational valued. The same applies to the 'constants' (defined by technology). Such problems are naturally algorithmic – i.e., procedural. It is this aspect that is highlighted in the key result I have called *Bharadwaj's Theorem I*, in the next section.

Bharadwaj noted, in the concluding lines of her review of *PCMC* (Bharadwaj, 1963, p. 1454; italics added):

> "Written in an unusually compact style and embellished with chiseled logic, [*PCMC*} bears the imprint of sustained reflection. Unmistakably, this is the work of a master written with authority and insight. …. [T]ime has dealt kindly with Sraffa's contribution. *It is as relevant today as it was when conceived* [over seven and a half decades ago]."

No one understood better than Krishna Bhardwaj that the 'chiselled logic' embodied in the theoretical propositions of *PCMC*, *proved with impeccably faultless mathematical reasoning*, albeit unconventional[4], were motivated by intensely *applicable* and *observational* relevance.

§2. Two Bharadwaj *Theorems*

> "Methodologically, [Sraffa]explicitly states that his immediate concern is the properties of the system which do not depend upon change. *Erroneously* interpreting this as invariance to change or changelessness, some have regarded Sraffa's analysis as restricted to a stationary equilibrium or, when extended, to signify steady states. There have been repeated charges about the ahistoricty of the exercise, which *appears as merely abstraction-mongering*."
> Bhardawaj, 1989, p. 321, italics added.

Neither Sraffa, nor Bharadwaj, stated any of their results or propositions in terms of the formal notion of a *Theorem*, although that is what they should have been called, given the pseudo-mathematical jargon of economic theoretical practice. Not calling them *theorems* has led various economists, with only a modicum of formal mathematical training, to carp and find lacunae in Sraffa's rigorously

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[4] I have maintained, ever since I noted it in my review of Pasinetti's *Lectures on the Theory of Production* (Velupillai, 1980), that *every* proved proposition in *PCMC* is mathematically rigorous and all attempts at recasting the formalism employed in the book in terms of linear algebra are unnecessary (Velupillai, 2008). In particular, it is not at all necessary to invoke the celebrated results of Perron & Frobenius to 'prove' Sraffa's 'theorems'.

derived demonstrations of his exact, impeccably unambigouous, propositions. Even worse misunderstandings have resulted in Sraffa avoiding the use of the word 'proof' to indicate the rigorous procedures with which he demonstrates the validity of his propositions (i.e., Theorems)[5].

Krishna Bharadwaj followed Sraffa's noble example and did not refer to her results and propositions, particularly in Bharadwaj (1970) as *theorems*; nor did she allude to her *procedural demonstrations* as *proofs*. In what can only be termed a truly prescient and remarkable, footnote, Krishna Bharadwaj pointed out (ibid, footnote 13, p. 415; bold italics added):

> "Analytically there is **no loss of generality** involved in a **procedure of successive consideration of production systems** using a different method of production for only one of the commodities common to them as, given all possible systems of production, **it could not lead to any different outermost boundary of wage-profit curves**. Incidentally, it would be noted that whatever be the number of commodities produced by different methods in the two systems the *maximum* number of switching possibilities would still be equal to the total number of distinct (without double counting) basics in the two systems together."

In the language of conventional mathematical economics there is (at least) one formal theorem in this observation – that which is gave the paper its title '*The Maximum Number of Switches Between Two Production Systems*'. More importantly, there is also a clear hint on stating, as a theorem, a result on the *uniqueness* of the 'outermost boundary of the wage-profit curves', but that is not all. There is also a clear statement of a way to use – a mode of constructing a procedure - this unique outermost boundary of the wage-profit curves' in comparative studies between production systems, *algorithmically*. The precise computational complexity of the procedure is, of course, not mentioned, but such things did not exercise the mind or the pen of many economists then, and they don't do so even now, despite much hype about computable general equilibrium theory (even in developmental contexts) and computational economics.

The above observations are summarized in three numbered statements, in the concluding *section iv* of this classic paper, simply referred to in terms of the phrase, '*To sum up*'. They encapsulate, again in the jargon of the more formal, if less justified, mathematical economist two absolutely fundamental theorems, both of which have important applicable richness. To state them more formally, giving them the completely justified prefix 'Bharadwaj':

> <u>*Bharadwaj's Theorem I*</u> (*ibid*, pp. 423-4):
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> At a switch point the adjacent production systems differ in the method of production for only one of the commodities common to them. The maximum number of switching possibilities between two such systems is equal to the number of distinct (i.e., without double counting) commodities entering, directly or indirectly, into the two alternative methods which respectively characterize the two systems.

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[5] Two unfortunate examples of this mischief by so-called mathematical economists are those by Burmeister (1965) and Hahn (1985), as I have pointed out in Velupillai (2008).

Bharadwaj's Theorem II (*ibid*, p. 424):
The choice of the value unit does not affect the maximum number of switching possibilities.

If I succumb to the temptation of pseudo-mathematical practice, then I would add what those who indulge in such mumbo-jumbo call a *Remark*:

Remark

The economic content of *Bharadwaj's Theorem II* is simply that the choice of the *numeraire* does not affect the content of *Bharadwaj's Theorem I*.

As for *Bharadwaj's Theorem I*, it may be useful to recall Sraffa's remark (sic!) on the different way orthodox, marginal, theory interprets and utilizes the existence of switch points, as clearly brought out in Kurz (op.cit, p. 4):

"The characteristic feature of switchpoints is that both distributive variables, wages, *w*, and the rate of profits, *r*, are rigidly fixed and are the same in both systems. Sraffa found this marginalist presupposition unacceptable. In a note written on 15 December 1943 he stressed that:

[T]he so-called determination is due to circumstances, which exist, not in the real world of actual production, but only in the world of imagination and possibilities: they are not intrinsic to the [actual] system and other levels [of *w* and *r*] cannot be 'inconsistent' with it. As far as the real, existing, system described by the equations is concerned, any levels [of *w* and *r*] are consistent with it. (Sraffa Papers D3/12/35: 43(2))."

In other words, the orthodox – i.e., marginalist – mode of analyzing a given system of production is, counterfactually, to postulate the existence of hypothetically 'adjacent' – i.e., 'marginally different from' – production systems. The important qualification by Sraffa is that such orthodox counterfactual reasoning is valid 'only in the world of imagination and possibilities: they are not intrinsic to the [actual] system', i.e., in the world of *parables* and have nothing to do with *realism*.

Bharadwaj's Theorem I is entirely consistent with Sraffa's strictures and has nothing to do with counterfactual, hypothetical, 'adjacent' production systems.

§3. Tractably Computing Wage-Profit Frontiers Using the *Bharadwaj Theorems*

"If one measures labour and land by heads or acres the result has a definite meaning, subject to a margin of error: the margin is wide, but it is a question of degree. On the other hand *if you measure capital in tons the result is purely and simply nonsense. How many tons is, e.g., a railway tunnel?*"[6]
Sraffa (1936); italics added.

Some twenty years ago I noted that[7]:

[6] The quote goes on (italics added):
"If you are not convinced, try it on someone who has *not been entirely debauched by economics*. Tell your gardener that a farmer has 200 acres or employs 10 men – will he not have a pretty accurate idea of the quantities of land & labour? Now tell him that he *employs 500 tons of capital*, & he will think you are dotty – (not more so, however, than Sidgwick or Marshall)."

> "Production structures carry with them natural prices corresponding to particular analytical assumption about the economics of the production system. What is needed is a device for extracting these prices from the observed data of a functioning economy."

Now, twenty – or so – years later, Stefano Zambelli has developed an algorithm to compute the index I constructed, based on wage-profit curves, to compare production systems, to measure technological progress. The connection with the Bharadwaj Theorems is the relevant point in the implementation of the algorithms. Briefly, so-called 'brute-force' algorithms are of exponential time-complexity, as the methods of production, for any production system, is increased in number. Using the two Bharadwaj Theorems, the exponential-time computational complexity can be reduced to polynomial-time complexity in increasing the number of methods of production.

Outline of the brute-force algorithm (Zambelli & Fredhom, 2010):

1. Organise input data – for example standard Input-Output tables – in terms of multi-dimensinal arrays (say using Matlab).
2. Enumerate all possible combinations of methods of production.
3. Compute, sequentially, the wage-profit frontier for each of the combinations of methods of production.
4. Retain the dominating value of wages (compared to its value for the preciously computed wage-profit frontiers).

The proverbial 'curse of dimensionality' enters copiously in the above algorithm because, for each rate of profit, all possible combinations of methods of production have to be evaluated. For example, for N countries, each with n production sectors, the total number of wage-profit frontiers are N^n. In the applications in Zambelli & Fredholm (op.cit), $N = 64$ and $n = 23$, which implies $N^n = 64^{23} \approx 3.5 \times 10^{41}$. A standard computer, running one whole year, must evaluate 1.1×10^{34} wage-profit frontiers per second. No computing facilities existing in standard format or at 'normal' institutions have the capacity to do any such computing today – or in the reasonable future.

This is where the *Bharadwaj Theorems* enter and help dissipate the 'curse of dimenionality': if the computational complexity measure of a brute-force algorithm is given by S, then utilizing the *Bharadwaj Theorems* this measure can be reduced to the order of $S^{1/n}$.

I shall not enter into technical details of the algorithm that has been devised, using the *Bharadwaj Theorems*, in this paper (the interested reader can see the details in Velupillai, 1994 and Zambelli & Fredholm, 2010). The point I wish to make, in this *homage* to the memory of Krishna Bharadwaj, is

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[7] In a paper prepared for the World Bank, with the collaboration of Stefano Zambelli (Velupillai, 1994). The paper had the 'tragic' fate of being approved for publication as a World Bank Discussion paper (on the basis of one supportive referee's report, by Lance Taylor, although heavily 'denounced' by the second referee, T.N. Srinivasan), and then the commitment to publish it in that mode completely ignored!

the following. Theoretical results, entirely motivated by empirical anchorings, devised by Krishna Bharadwaj, based on the framework developed by Sraffa in *PCMC*, suggested an algorithmic procedure for computing wage-profit frontiers. These, in turn, were the basis on which I constructed an index, which now can be computed with measurable computational efficiency, as a result of Zambelli's understanding of the relevance of the *Bharadwaj Theorems* for this purpose.

Surely, this is an effective counter-example to the pointless and ceaseless accusations of abstraction-mongering and ahistoricity of the Sraffian framework? And it was achieved by a combination of theoretical ingenuity on the part of Krishna Bhardwaj, intuitive knowledge of the procedural nature of the demonstrations of propositions in *PCMC* by Bharadwaj, felicitously combined with Zambelli's desire to circumvent the neoclassical counterfactual fudge of productivity comparisons using untenable production functions, having as arguments non-measurable inputs.

§4. Brief Reflections of Krishna Bharadwaj's Sraffian Methodology

> "[T]he review, 'Value through Exogenous Distribution,' appeared in August 1963. In order to write it, she followed a demanding intellectual pilgrim's progress, taking the same journey that Sraffa himself had taken over the 30 to 40 years prior to the publication of the book. She read, as he did, Smith, Ricardo, Malthus, Marx, Mill, Jevons, Marshall, Walras, Wicksell. The result was her outstanding review article …"
> Harcourt, op.cit, p. 301.

I believe the Sraffian basis of Krishna Bharadwaj's scientific methodology is most clearly evident in Bharadwaj (1970). In particular, in her mode of discussing the nature of the economically motivated difference between the Sraffian categories of *basic* and *non-basic* commodities and the mathematical distinction between *decomposable* and *indecomposable* matrices. This difference, when obfuscated by ill-digested mathematics, grafted on to an economic theory and its categories without serious anchoring in observable relevance, grounded in the traditions of economic thought, leads to the kind of appeals made to 'aggregation' by Peter Newman (op.cit) and Levhari's hasty conclusions (Levhari, 1965). Her concluding remark on this double distinction are worth their economic weight in gold (*ibid*, p. 423):

> "The classification of commodities into basics and nonbasics in a given system uses more of the available information about the system than does the classification of the system as decomposable or indecomposable. The additional information incorporated in the former distinction is essential for the discussion of switching possibilities between two systems."

It is not surprising, then, that it was Krishna Bharadwaj who derived the important two Theorems that enabled us to construct an algorithm and compute efficiently an intractable measure and use it for the eminently empirical purpose of comparing the productive capabilities of two economic systems.

Krishna Bharadwaj had an admirably complete command of the traditions of economic thought that underpinned much of contemporary economic theory. She, like Sraffa, may not have had command over the formal side of mathematics, but had an unsurpassed intuition to guide her towards an understanding of the way economic theory had to be developed to serve in the interpretation of the evolution of economic institutions – and how they might be shaped to further humane ends, in their future development.

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