Zermelo, Euwe and König on Set Theoretic Chess

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Abstract

Ernst Zermelo stated two ‘theorem(s)’ on chess, and proved one of them, in 1913. Thirteen years later, in 1926, Dénes König proved the other, inducing also Zermelo to do the same. Both developed the theorems and proofs within classical set theory, with the axiom of choice (ZFC - Zermelo-Fraenkel set theory with the Axiom of Choice). Max Euwe, in 1929, based his results on chess in Brouwer’s (intuitionistic) set theory. The implications of these alternative approaches, especially on proofs, is discussed in this paper.

JEL Codes: B16, C02, C62, C70, C79

Key Words: Zermelo, Euwe, Chess, Games, Set, Proof

†Zermelo and König develop their results - theorems and proofs - with the full ‘power’ of classical set theory, i.e., within Zermelo-Fraenkel set theory, including the axiom of choice (ZFC) & the tertium non datur; Euwe’s results are based on Brouwer’s intuitionistic-constructive set theory. That (the axiom of) choice implies the tertium non datur is a later ‘formal’ result of Bishop (1967); Diaconescu (1975); Goodman and Myhill (1979), although we think it was known to Brouwer from at least 1910.

‡Tottvägen 11, 169 54 Solna, Sweden. kvelupillai@gmail.com. We depart from the usual convention of alphabetical ordering of authors for similar reasons as mentioned in the preface of Zadeh and Desoer (1974). However, we are less than convinced about the use of a ‘fair’ coin to decide in the light of experimental results in Diaconis et al. (2007), which casts serious doubts on the ‘fairness’ of the exercise.

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1 Preliminaries

"[W]e at this congress\textsuperscript{1} have a peculiar responsibility to see that [Brouwer’s] thought does not die with him.\textsuperscript{2} Myhill (1968, p. 167; italics added)

Chess is a finite game, between two players - say, white and black or 0 and 1 - with diametrically opposed interests, who have complete information about past plays i.e., moves - and with no aspect of chance\textsuperscript{3} involved in the enunciation of its rules of playing (where, by convention, white plays first), or in its results (Shenk, 2007). They are also (often) referred to as win-lose deterministic games with perfect information (or perfect recall by the two players).

Chess is played on two-dimensional boards with a prescribed number of squares, given as $8 \times 8$, at the outset. The movable pieces in chess - pawns, knights, bishops, rooks, queen and king - are finite in number, and so are the squares on which they can be placed and the moves\textsuperscript{4} are constrained by rules. König (1927, see below) does consider an infinite board - basically to be able to apply his infinitary lemma (again, see below), but constrain the pieces to take positions only in those prescribed by the rules that apply to the $8 \times 8$ board (i.e., 64 squares and rules of replacement of the pieces, castling, etc.).

For conciseness, we shall refer to such Chess games as Zermelo-type games or Zermelo Games, ZG (somewhat motivated by Zermelo-Fraenkel Set Theory with the Axiom of Choice as ZFC).

It is, in contrast to classical game theory, a Combinatorial Game - more particularly, a Partizan Game (where the Sprague-Grundy theorem of Combinatorial Game Theory - i.e., first-player move advantage - is inapplicable). These games, like their counterparts in classical game theory by von Neumann, von Neumann & Morgenstern, Shapley, etc., were developed in the early 20th century. Due to Conway (1976), they are linked to advances in number theory.

On the other hand, chess can also be viewed as a branch of Arithmetical Games, eschewing the axiom of choice and adopting the axiom of deter-

\textsuperscript{1}The 3\textsuperscript{rd} International Congress for Logic, Methodology and Philosophy of Science, held on August, 25 - September 2, 1967, in Amsterdam (Van Rootselaar & Staal, 1968).

\textsuperscript{2}Brouwer died (in a traffic accident) on 2\textsuperscript{nd} December, 1966. He was, by the way, ‘for some time the president of the chess club of Blaricum. . . [and] a fervent chess player.’ (van Dalen, 2013, p. 643).

\textsuperscript{3}We prefer to refer to this as non-determinism, for computational complexity theoretic reasons.

\textsuperscript{4}The number of moves depend on the player whose turn it is to play, which is, in turn, a ‘function’ of the kind of piece that is moved.
minateness (Mycielski (1964); Steinhaus (1965); Takeuti (2003), especially §5.1), and being a part of the advances in the foundations of mathematics contributing to games of strategy.

Apart from the term finite, all the other italicized words are accepted, and used, by Zermelo, König and Euwe\(^5\) in a common sense vein. Finite - and its ‘antonym’, infinite - are defined, or used, in the context of the kind of set theory that each of the protagonists work with - i.e., classical or constructive set theory.

The paper is organized as follows: the next section concentrates on the pioneering contributions of Zermelo - the statement of his theorems, the analytical content of the proofs of these theorems, the stated and unstated assumptions and the undefined terms. In §3 and §4, we concentrate on the contributions of König and Euwe, from similar standpoints. The concluding §5 is confined to a definition, a lemma, some remarks and a theorem on the complexity class of ZG, together with some ‘critical’ observations on von Neumann.

This paper emphasizes - at least in the next three sections - the underpinnings of theorems and proofs in classical and constructive set theory, with and without the assumption of ZFC and the tertium non datur, by Zermelo, König and Euwe.\(^6\) In this sense only, we think that Schwalbe and Walker (op.cit) - in an otherwise admirable paper, with which we agree for the most part - are mistaken. In their concluding lines about the future of the research agenda that Zermelo, König and Kalmár initiated\(^7\), they state (ibid, p. 133; italics added):

The concerns of Zermelo, König, and Kalmár were answered

\(^5\)We do not discuss Kalmár (1928/1929) or von Neumann (1928). The former is - in our opinion - adequately and competently analysed in (Schwalbe and Walker, 2001, see especially p.124, footnote 6 & §4, p. 7 ff), but also because of the use of ‘free-swinging’ Cantor-type arguments invoking the properties, explicitly, of infinite and thoroughly non-constructive infinite ordinals and ordinary (infinite) cardinals (which are implicitly and explicitly used also by Zermelo and König). But we think, on the basis of previous studies by us of Kalmár’s important writings, that he was aware of such distinctions - which was, clearly, not the case with König (see, in particular, (Franchella, 1997, pp. 24-25). As for Zermelo, we prefer to be agnostic, especially on this topic! In the case of the latter, one of the only disagreements (see below for the ‘other’) with Schwalbe and Walker (2001) - a trivial one - is with the translation of the title of von Neumann’s 1928 paper (ibid, pp. 132/3). We almost equally trivially disagree with the translation from the original German, of phrases and words, into English, in Zermelo (1913; 2010). These translations - for example, Verstandesspiele as games of reason - do not, in our opinion, mar their otherwise impeccable classical analysis of the determined game.

\(^6\) All of whom, and Kalmár, work within ZG.

\(^7\) And, in a different, intuitive-constructive sense, initiated in a (partly) still-born sense, by Euwe (but see, (Elkies, 1996, p. 135).
at a very high level of generality in the paper by Kalmár and thus have not generated an ongoing research agenda.

But they are, of course, correct when they also state, in the previous sentence to the one above (italics added):

“In contrast to the work of Zermelo, König, and Kalmár, von Neumann’s main concerns were the strategic interaction between players and the concept of an equilibrium. These two ideas have become the building blocks of modern non-cooperative game theory.”

2 Zermelo’s Theorem(s) & Proofs

“Now for me, all theorems that one states about finite numbers are in the end nothing but theorems about finite sets. Therefore, it is necessary at the outset to define what is meant by that.”

(Zermelo, 1909; 2010, p. 237; italics added)

In a comprehensive technical and largely complete historical introduction to Zermelo (1913; 2010), Larson (2010, p. 265; bold italics added)8 reinforces the standard view on references to this classic:

Aside from the work of König and Kalmar, the earliest9 citation of Zermelo’s 1913 that we have been able to find is Kuhn’s 1953.

In our view, Zermelo (1913; 2010) states the following two theorems on ZG10:

1. There are classically provable mathematical necessary and sufficient conditions for one of the two players, in any of the given finite positions, to force a win, in at most a given number of moves;

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8Larson notes (ibid, p. 263) that ‘König credits this application of his lemma to a suggestion of von Neumann.’ to clarifying some of the ‘looseness’ in Zermelo (1913). For a clarification of the nature of this ‘lemma’ in König (1926), as suggested in Larson (ibid), see Franchella (1997, p. 25, italics added), ‘In the same year [1926], he [König] isolated a part of the proof as a lemma apart.’


10Ewerhart (2002, p.206-7) claims that Zermelo’s paper ‘does not make a formal statement’ of the theorem and that it was Kalmár who presents a formal statement. However, this claim cannot be evaluated without a precise does not define of the term ‘formal’.
2. There exists a classically provable upper bound for the number of moves in which one of the (two) players can force a win;

The first of the ‘theorems’ was proved in Zermelo (ibid); the proof of the second, by Zermelo, was ‘about’ 14 years later, as a result of dissatisfaction with the details of the proof, with a ‘gentle prodding’ query, by Dénes König in 1927 (cf., §5, in König, 1927). Zermelo’s results, together with that part of König (ibid, §5) are excellently presented and discussed in Schwalbe and Walker (2001, especially, p. 126, ff) and we direct the interested reader to it (and, of course, Zermelo’s originals - but also to König, ibid).

However, Zermelo’s proofs of the two ‘theorems’ are:

- Proved indirectly, i.e., using the method of reductio ad absurdum (as also pointed out by Schwalbe and Walker, 2001, pp. 126/7 - in the former page they refer to it as ‘proof by contradiction’);

- Invoking ‘indiscriminately’ the axiom of choice in the proofs, in the sense that Zermelo seems not to know when, and where, this axiom is used in his proofs (as pointed out by König, too, in the second sentence of the last paragraph of §5 of his infinitary lemma paper of 1927).

Two further remarks on Zermelo (ibid) may well be appropriate here, but some description of the notations in them are mildly necessary. Firstly, Zermelo considers finite positions, for the finite # of pieces, \( p_i \), \( i = 1, 2, \ldots \), \( t \), in any play of \( \text{ZG} \). Secondly, at any point in time, of the play of a \( \text{ZG} \), any of the finite # of pieces (pawn, rook, knight, bishop, queen & king), occupying one of the finite # of squares (64 in the case of standard chess board), is assumed to have a finite or infinite continuation, \( q = q_1, q_2, \ldots, \ldots \). This entails, at a trivial level, no assumption(s) on stopping rules\(^{11}\) for a play in \( \text{ZG} \), but at a ‘deeper’ level, invoking of some form of the axiom of choice in his proofs of the two ‘theorems’ by Zermelo.

In concluding this sub-section on Zermelo, we should point out that in an intuitionistic constructive proof, particularly of a Brouwer-type, reductio ad absurdum, in its ‘normal’ senses, has no place; but, as Berg et al. (1975, p. 226; italics added):\(^{11}\)

\(^{11}\)See Schwalbe and Walker (2001, p. 125 & footnote 9 on the same page). Larson (2010, p. 260, footnote 2 & p. 261, footnote 3) cannot be correct. Just including ‘the list of previous moves’ does not make the finiteness assumption for \( p_i \) invalid; and the relevance of the axiom of choice, for Zermelo’s proofs, are not diminished by the \( p_i \) being finite. Also, the remarks on quasi-strategy and the reference to lKechris (1995), in our opinion, are gratuitous.
“Although we cannot in general argue by reductio ad absurdum, we can use this method of proof in the following way. If A and B are the only possible outcomes of a finite computation, then we can demonstrate that A occurs by showing that the occurrence of B leads to a contradiction.”

We will have more to say on this in sections 4 & 5.

Finally, König (1927, p. 130, footnote 10) does indicate that ‘Mr. Zermelo intends to publish a comprehensive account of his investigations in the theory of chess in the near future.’\textsuperscript{12} - although to the best of our knowledge such has not been the case.

### 3 König’s Infinitary Lemma & Proofs of the Zermelo Theorem(s)

In the graphical interpretation of graph theory, at least in what concerns finite graphs, [the first] chapter contains for the most part only the obvious. . . . In what, in particular, concerns infinite graphs I could cite only my own works. König (1936; 1990, p. 49; italics added.)

Tutte (1990, pp., 24/6; italics added), in his excellent and comprehensive ‘commentary’ on König (1936; 1990), summarises the import of the unendlichkeitslemma, the infinity lemma, lynchpin of König’s applications\textsuperscript{13} to ZG:

“[T]he ‘Unendlichkeitslemma’, [is] a device by which many theorems already proved for finite graphs can be extended to infinite but locally finite ones. . . . The Unendlichkeitslemma has been a powerful tool for investigating locally finite graphs. . . . . In Chapter VIII . . . the application to Game Theory is discussed in some detail. . . . [W]e can consider that the game [ZG] is played on a directed graph G. . . . The directed edges are the moves; each is directed from the initial position of its move and to the resultant one. In theory it is possible to classify each vertex as (i) a winning position for the mover, (ii) a losing position for him or (iii) a

\textsuperscript{12}The original German is: “Herr Zermelo beabsichtigt seine Untersuchungen über die Schach-Theorie demnächst in zusammenhängender Darstellung erscheinen lassen.” The English translation above, is in Zermelo (1927; 2010, 2010, p. 351, footnote 2).

\textsuperscript{13}At the suggestion of von Neumann (König, 1927, p. 125).
drawn position. . . . Other board games can be represented by directed graphs . . . , chess for example.”

Hence:

**Definition 1.** Infinite, locally finite, (directed) graphs are equivalent to trees.

**Definition 2.** ZG, as directed graphs, are trees.

**Definition 3.** Every move in a ZG is a branch of the tree representation of the directed graph that is the ZG.

**Theorem 1.** König’s Infinitary Lemma for Trees, (Kuratowski and Mostowski, 1976, p. 326).

If T is an infinite tree with finite levels [degrees] which for each integer n has chains\(^{14}\) with at least n elements, then T has an infinite chain.

**Proof.** See Kuratowski and Mostowski, 1976, pp. 326-7. ■

**Remark 1.** The axiom of choice is explicitly invoked in the proof, through the use of a Choice Function\(^{15}\).

All of the proofs, on applications of the infinitary lemma - especially in ZGs - in König (1927, 1936; 1990) are based on an invoking of the axiom of choice and the use of reductio ad absurdum method of proof.\(^{16}\) This is made abundantly clear in Franchella (1997, p. 26; italics added):

We can add that in the case infinity lemma, Dénes’ scant philosophical attitude was revealed by his not very deep analysis of the role of the axiom of choice in its proofs. . . . [H]e did not seem to have noticed this . . . although he had promised . . . not to use the well-ordering principle\(^{17}\) [which is equivalent to the axiom of choice] . . . in his proofs.\(^{18}\)

\(^{14}\)Suppose \(X\) is a set, ordered by the relation \(\preceq\); and if \(Y \subseteq X\), where for any two elements of \(Y\), say \(x \& y, x \preceq y\) or \(y \preceq x\), then \(Y\) is a chain in \(X\) (Kuratowski and Mostowski, 1976, p. 80).

\(^{15}\)See the definition of a Choice Function used in theorem 8, Kuratowski and Mostowski, ibid, p. 73. As Kuratowski and Mostowski write in the ‘Preface to the First Edition’, p. vi: “In order to illustrate the role of the axiom of choice we marked by a small circle ° all theorems in which this axiom is used.” Theorem 8 is so marked!

\(^{16}\)For, the third (of four, cf., also Franchella (1997, p. 24) and Kuratowski and Mostowski (1976, p. 328, exercise 6)) application of the infinitary lemma to be valid, König’s assumptions are insufficient (cf. Wolf, 2005, p. 38).

\(^{17}\)As Zermelo called it.

\(^{18}\)On the use of the method of reductio ad absurdum, in the proof of the infinitary lemma, see for example, p. 21 & footnote 19, on p. 22, of Franchella’s elegant and brilliant article.
Konig (1936; 1990, p. 171, footnote 4) himself\(^{19}\) ‘confesses’ that, in the proof of the infinitary lemma, the axiom of choice is used (indiscriminately, we would like to add, not with the care that is evident in Kuratowski and Mostowski, 1976). He added, somewhat mysteriously (in the same footnote, italics added):

> With most applications of the Infinity Lemma this axiom can nevertheless be avoided, into the details of which we shall not go.

We would like to end this section *respectfully disagreeing* with Konig that ‘[the] axiom [of choice] can nevertheless be avoided, in ‘most applications of the infinity lemma’. We do not think that the ‘infinity lemma’ can be proved without invoking the *tertium non datur* and, therefore, the axiom of choice and conventional proof by contradiction. A comparison of Konig’s enunciation of the infinity lemma, and it’s proof, in Konig (1936; 1990), ch. VI, §2, with one of Weierstrass’ original statement of what came to be known as the *Bolzano-Weierstrass Theorem* (Moore, 2008, p.222), makes this abundantly clear.\(^{20}\)

### 4 Max Euwe’s Intuitionistic-Constructive Theorem & Proofs of ZG

The set S has a subset E of *finite* chess matches. .... The subset U of *infinite* chess matches likewise shares the same law of generation as S with one addition: the law of creation of a configuration without choice or one that inevitably leads to a configuration without choice, results in inhibition and destruction. Euwe (2016/1929, p.12)\(^{21}\)

A brief remark on Euwe’s professional chess career (for details see Munnighoff, 2007), and credentials as a competent player of the game, may not be out of place here, especially to understand why he may have chosen ZG as a repository for the application of Brouwerian intuitively constructive set theory. Euwe became the World Chess champion, defeating Alexander Alekhine\(^{22}\).

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\(^{19}\)See also footnote 8, p. 78 - where, by the way, the well-ordering theorem is referred to as ‘Zermelo’s Axiom of Choice’.

\(^{20}\)In this connection, the two essays by Brouwer (1952 A, pp. 506-7 & 1952 C, pp. 516-518), in Brouwer (1975) on an *Intuitionistic* re-formulation of the *Bolzano-Weierstrass theorem* are particularly relevant.

\(^{21}\)The italics on ‘finite’ and ‘infinite’, added.

\(^{22}\)Shenk (2007) describes, colourfully, some of the unsavoury aspects of the life of Alekhine.
on December, 15, 1935. His reign lasted for two years, before Alekhine defeated him, again, in 1937. He authored, or co-authored, many books on chess playing and strategies.

The title of Zermelo's classic of 1913, on what we refer to as ZG, is: Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels. The title of Euwe’s Brouwerian intuitionistic constructive analysis of the playing of the game of chess is: Mengentheoretische Betrachtungen über das Schachspiel.23 Without an understanding of the precisely different mathematical foundations - the metamathematics - of the two notions, the reader of the latter would not have a clue as to the nature of the min-max derived, with precisely exact numerical approximation, by Euwe.

At the minimum, a sympathetic reader of Euwe’s classic, will need to acquaint herself with the rudiments of Brouwerian Intuitionistic Constructive mathematics and its metamathematics.24 This means, at least to read this paper by Euwe, with some mathematical competence, some minimal acquaintance with the precise constructive intuitionistic notions of the ‘triptych’ of:

- Sets
- Spreads
- Proof

Ideally, a reader of this classic by Euwe should have come to it without any preconceptions of mathematics and its foundations; not with any acquaintance with any kind of alternative game theoretic formalism, especially ‘standard’ ZG formalism. Even if the former is within the realm of possibility, the latter prerequisite is quite unlikely.

In §1, we distinguished between: the von Neumann-Morgenstern conventional - at least in the social sciences, especially in varieties of economic and applied economic analysis; combinatorial game theory and arithmetical games. In all of these some form of the axiom of choice or the axiom of determinateness, together with reductio ad absurdum method of proof, is invoked and used (especially in proofs of theorems belonging to these fields). To these three, we must add Brouwerian Intuitionistic Constructive Game Theory, which is attributable to the single, outstanding, contribution by Max Euwe. In the proofs of theorems, Euwe, true to the tenets of Brouwerian Constructive Intuitionistic mathematics, does not invoke any of the axioms

23In both cases the boldface emphasis has been added.
24For more on this, see Dummett (2000); Troelstra and van Dalen (1988); Heyting (1971).
mentioned above, nor use the non-constructive versions of proof by *reductio ad absurdum*.

In van Dalen’s admirably complete biography of Brouwer (van Dalen, 2005, p. 636; italics added) it is stated:

In 1929 there was another publication in the intuitionistic tradition: an *intuitionistic analysis of the game of chess* by Max Euwe [(1929)]. It was a paper in which the game was viewed as a *spread* (i.e. a tree with the various positions as nodes).\(^{25}\) Euwe carried out *precise constructive estimates* of various classes of games, and considered the influence of the rules for draws.

And in König (1936; 1990, p. 161, note 15; italics added), referring to Euwe (1929):

> From the standpoint of Brouwer’s *intuitionism* M. Euwe has concerned himself with similar questions.

‘Similar questions’ in the sense of the Theorems and their Proofs in *ZG*, but from the standpoint of intuitionistic set theory and constructive methods of proofs, without any assumptions of axioms or the use of *reductio ad absurdum*.\(^{26}\)

### 5 Concluding Notes

*Modern mathematics* is nearly characterized by the use of rigorous\(^{27}\) proofs. Jaffe and Quinn, 1993, p. 1; italics added.

\(^{25}\)This is the *Brouwerian constructive* mathematical equivalent of the standard formulation of extensive form games.

\(^{26}\)See also Ebbinghaus and Peckhaus, 2007, p. 144, where they write (referring to Euwe (1929), over 70 years after König (1936; 1990) and 2 years after van Dalen (2005)):

> “Subsequently, without knowing the anticipations of Zermelo and König, Max Euwe, one of the most influential chess players of his time and world champion from 1935 to 1937, *independently published parts of the results in an intuitionistic framework*.”

Although Euwe’s article was published in 1929, we conjecture that he worked in getting the result *at least* from 1927 (van Dalen, 2013, p. 472) - the year in which König published the application of the infinitary lemma to *ZG* (at the suggestion of von Neumann, cf. §5, below, too).

\(^{27}\)We are not sure, in the absence of something like a ‘Church-Turing Thesis’, how unambiguously this term can be defined and used, especially in mathematical contexts.
We are of the opinion, substantiated in this essay (we hope!), that Zermelo, König (Kalmár) and Euwe, tried, with varying successes, to understand the playing of chess with mathematical tools and concepts they had developed independently of the game - and make the playing of the game ‘nearly’ rigorous. They did not, and could not, achieve complete success in making their (mathematical) proofs rigorous.

A definition, a lemma, five remarks and a theorem, together with a comment on the claims on algorithmic formulations by Zermelo, and some observations on von Neumann conclude this paper. The proof of the theorem is based on a ‘counting argument’, of finite cases of evaluating quantifiers.

Definition 4. (Wolf, 2005, p. 38) A first-order formula is in prenex normal form if it consists of a (finite) string of alternating existential and universal - i.e., ∃ & ∀- quantifiers, followed by a quantifier-free sub-formula.

Remark 2. Every movable piece - i.e., pawns, rooks, knights, bishops, queens and kings - are given integer-number values.

Remark 3. Every ZG statement is a first-order formula. Every evaluation function, for any ZG, ends with a quantifier-free sub-formula to decide a winner (or declare a draw).

Remark 4. Every node - finite in number in any given ZG - of the tree representation of any ZG is, alternatively (depending on whether it is white or black’s turn to play) one of the two quantifiers, ∃ or ∀.

Lemma 1. The tree version of any ZG can be transformed to Prenex Normal Form of propositional logic.

Proof. By construction - i.e., without invoking either the axiom of choice, or the axiom of determinateness, nor using as a proof method any classical form of reductio ad absurdum. ■

Remark 5. Like Fig. 46, on p. 201, in König (1936; 1990), referring to drawing directed graphs in solitaire games, it is ‘difficult’ - most likely as difficult as the Travelling Salesperson’s Problem - to draw any tree of a ‘realistically’ sized ZG. In other words, this must be a problem that can be classified as NP-Complete.

Hence:

Theorem 2. Every ZG in the Prenex Normal Form (of propositional logic) is NP-Complete.
Proof. Again, by construction. At every node of the tree, once converted - in principle, remembering that in any ZG one works with a finite number of nodes - into prenex normal form, either an existential quantifier, or a universal one, is evaluated. But the number of nodes increases exponentially, as the ZG ‘progresses’.

Remark 6. This is a purely theoretical result; it does not imply that particular ZGs are not NP-Complete (as in the case of the negative solution of Hilbert’s 10th Problem (cf., Matiyasevich, 1993). It is in the spirit in which Zermelo’s result(s) were obtained in his pioneering papers of 1912/1913 and 1927, enunciated as Zermelo (1913; 2010, p. 267; italics added):

[W]e shall not be concerned with any method for actually playing the game . . . . It certainly seems worthwhile to me to investigate whether it is at least theoretically possible, and whether it makes sense at all, to evaluate a position also in cases where it is practically impossible to carry out a precise analysis on account of the unsurveyable complication posed by the possible continuations of the game.

Remark 7. Various kinds of pruning - i.e., α & β pruning, practiced by good Chess players - can make an NP-Complete Problem solvable (for example, in polynomial time), or any ZG also practically solvable.

Under the conditions and result of Theorem 2 (and, in particular, ‘Remark 4’), the observation by Osborne and Rubinstein (1994, p. 6; italics added) is, at the least, questionable, especially because Zermelo did not define an algorithm for the solution of his two theorems (cf. §2, above - nor anyone for any ZG):

[C]hess is a trivial game for ‘rational’ players: an algorithm exists that can be used to ‘solve’ the game. This algorithm defines a pair of strategies, one for each player, that leads to an ‘equilibrium’ outcome . . . . The existence of such strategies suggests that chess is uninteresting because it has only one possible outcome. . . . Its equilibrium outcome is yet to be calculated; currently it is impossible to do so using the algorithm.

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28 That it is not a ‘trivial game’ is one of the key result of Brouwerian intuitionistic constructive mathematics and recursion theory (as clarified by Harrop’s Theorem, as in Harrop (1951)). It is given a ‘precise answer’ in classical game theory only because mathematical induction - or, what Brouwer would refer to as ‘complete induction’ - is unreservedly used in all known proofs of Zermelo’s Theorem on Chess (with the notable exception, of course, of Euwe). Brouwer’s negative answer - akin to Harrop’s Theorem - relies on Bar Induction (see below and also Dresden (1924) - especially pp. 37-40).

29 See also Schwalbe and Walker (2001, pp.132/3).
We think it is apt to quote Gary Kasparov’s poignant assertion at this point (with which we agree):

Chess is old enough for its origins to be less than entirely clear.


The ‘other side of the coin’ of chess ‘not being . . . just a game, is that it is always interesting.\(^{30}\)

We end this paper with some observations on von Neumann, prompted by the puzzled comments - in our opinion, of course - of Steinhaus, Euwe and van Dalen. To put Steinhaus’ observation in context, we begin with König’s expressly - and explicitly - stated indebtedness to von Neumann’s suggestion of the applicability of the infinitary lemma to ZGs. The earliest references by König are on pages 125 and footnote 9, of p. 129, in König (1927; italics added):

Die vierte Anwendung, die ich hier mitteilen will, verdanke ich Johann von Neumann und bezieht sich auf eine Art von Spielen zu denen auch das Schachspiel gehört.

And,

Laut früheren mündlichen Mitteilungen des Herrn J. von NEUMANN war ihm ein auf derselben Grundidee beruhender Beweis bekannt.

König also writes (Ebbinghaus and Peckhaus, 2007, p. 132; italics in the original), on 13 February, 1927:

I was the one who drew [von Neumann’s] attention to [Zermelo’s 1912/1913 paper]. Now he is working on a paper about games which will appear in the Mathematische Annalen.

\(^{30}\)Zermelo (1913; 2010, p. 273; italics added) in a ‘rare’ exaggeration, ends his pioneering contribution with the statement:

The question as to whether the initial position \(p_0\) already is a “winning position” for one of the players is still open. Its precise answer would of course deprive chess of its character as a game.

Of course, ‘game’ in the sense of a ZG.
Ebbinghaus and Peckhaus continue\(^{31}\) (italics in the original), after quoting the above letter from König:

This \([\text{letter}^{32}]\) leaves open several possibilities. Von Neumann may have started to think about games either by himself or may have been inspired by König who knew Zermelo’s paper and was working on related questions. One may also assume that it was Zermelo’s paper which prompted him to think about games more thoroughly. But von Neumann does not refer to it - neither in the *Gesellschaftsspiele paper* \([\text{von Neumann, 1928}]\) announced by König nor in the comprehensive book on game theory which he wrote together with Oskar Morgenstern . . . . To stay on stable grounds, we will therefore confine ourselves to giving Zermelo the credit for having written the first paper that mirrors the spirit of modern game theory.

Now, first, Steinhaus’s observations on von Neumann \((\text{Steinhaus, 1965, p. 460; italics added})\):

J. von Neumann was aware of the importance of the minimax principle \([\text{von Neumann, 1928}]\); it is, however, difficult to understand the absence of a quotation of Zermelo’s lecture in \([\text{von Neumann’s}]\) publications.

Next, van Dalen \((2005, \text{p. 636; italics added})\):

When \([\text{Euwe, 1929}]\) wrote his paper he was not aware of the earlier literature of Zermelo and Dénes König. *Von Neumann called his attention to these papers, and in a letter to Brouwer\(^{33}\)* Von Neumann sketched a classical approach to the mathematics of chess, pointing out that it would easily be constructivized.

Finally, a part \((\text{we shall return to the ‘other’ part, in the sequel})\) of the Euwe \((2016/1929, \text{p.19; italics added})\) observation:

After completing the preceding observations, I was *informed by Dr. Neumann about two other works dealing with the same topic* \([\text{those of Zermelo, op. cit., and König, 1927}]\).

\(^{31}\)But see Schwalbe and Walker \((2001, \text{pp. 132/3})\), which is more in consonant with our own views.

\(^{32}\)The original English word is ‘passage’.

\(^{33}\)Our ‘archival’ research, confined to a thorough reading of van Dalen \((2011)\), has, so far, not succeeded in locating this letter.
First we would like to emphasise an incontrovertible technical fact: von Neumann never constructivized the ‘classical approach to the mathematics of chess’ - nor is it true that ‘it would easily be constructivized’. In fact, to the best of our knowledge, von Neumann never ‘constructivized’ any proof of any of his game theoretic (or growth theoretic) theorems.

Secondly, Euwe’s ‘reactions’ to von Neumann’s remarks are worth quoting in full (Euwe, 1929, pp. 641-2; Euwe, 2016/1929, pp. 19-20; bold italics in the original):

In the addendum Zermelo .. proves that, if a victory can be forced in a given position, it has to be possible in a limited number of moves. Seeing as the first part of my Sec. 3 roughly corresponds to Zermelo’s remarks, I could have omitted this part of my paper. I have not done this because it is presented in a completely different form and because I wanted to examine the core of the solution the reduction to a finite problem more closely.

König defines the term ‘winning position’ and proves that, if it is possible to force a victory for White from a position \( q \), it also has to be possible to do so in fewer than \( N \) moves. (\( N \) is a finite number determined by \( q \)).

He thereby also defines the best possible move in a given position.

It is remarkable that the remarks by König remain valid when the set of all positions is no longer finite. (König provides the example of an infinite chessboard with the same move limitations as in a finite chessboard.).

However, the provided proof is not constructive, i.e. no method is shown by which a victorious move order, if at all possible, can be constructed in finite time. It is arguably possible to determine for each position of this generalized game of chess whether a checkmate is possible in \( n \) moves or not. Nevertheless, no upper bound exists for \( n \) [that can constructively be determined, using König’s infinitary lemma].

We remain as puzzled as Steinhaus at the absence of a reference to Zermelo (1913; 2010) or to any of König’s writings on the applications of the infinitary lemma to ZGs (especially of the 1920s or 1930s), in any of von Neumann’s influential writings on game theory, few as these were. But we would like to wonder, in the same spirit as Steinhaus, why von Neumann never referred to Euwe’s contribution of 1929. We conjecture that this was due to his lack of interest - certainly not due to a lack of competence - in proof methods of a Brouwerian intuitionistic constructive sort, till about the 1950s.
References


Tutte, W. T (1990), Commentary on König, Dénes (1936; 1990).


